Floer theory : Homework 1

Paramjit Singh

Terminology. Let X be a manifold with a Morse function f on it. We shall refer to $f(p)$ as the *height* of $p \in X$. By a *small* neighborhood around a critical point, we mean one which doesn't include any other critical point.

Problem 1. Let $\gamma : \mathbb{R} \to X$ be a gradient flow line. Show that $\lim_{t\to\pm\infty} \gamma(t)$ exist and are critical points.

Solution. Suffices to show $\lim_{t\to\infty} \gamma(t)$ exists and is a critical point (and then take negative of the Morse function for the other result).

Let t_n be a sequence such that $t_n \to \infty$ as $n \to \infty$. Suppose $\gamma(t_n)$ exists. Then it must be a critical point.

(Intuitively, if the limit is a regular point, then one can still flow along the direction of the gradient, and thereby reduce height, and hence we can't have $\lim_{n\to\infty} \gamma(t_n)$ to be a regular point, as height strictly decreases for $\gamma(t)$ as t increases.)

More precisely, if $p = \lim_{n\to\infty}$ is regular, it has a neighborhood of regular points around it, say W. By the flow box theorem, there exist neighborhoods U of p in $f^{-1}(f(p))$, and $(-\varepsilon, \varepsilon)$, such that the flow function $\varphi : U \times (-\varepsilon, \varepsilon) \to W$ is a smooth embedding (a diffeomorphism onto its image, which is open in W). Under this embedding, $\operatorname{grad} f$ is mapped to $\partial/\partial x$, which strictly moves $\vec{0}$, and hence the flow of $\operatorname{grad} f$ moves p. Thus $f(\varphi(p, \varepsilon/2)) < f(p)$ and hence if $t_{n+1} > t_n + \varepsilon/2$, $f(\gamma(t_{n+1})) \leq f(\varphi(p, \epsilon/2) < f(p)$, a contradiction. □

Now we show $\lim_{t\to\infty} \gamma(t)$ exists. As X is compact, $\{\gamma(t) : t \in \mathbb{R}_+\}$ does have limit points. Each of them must be a critical point, as shown above. Let p and q be two such distinct points.

Enclose all critical points in an open set U, comprising disjoint balls around each critical point. Then $X \setminus U$ being compact, $|\operatorname{grad} f|$ is bounded on $X \setminus U$ ($|\cdot|$ being taken with respect to a Riemannian metric on X).

Assume X is connected, and let d be the geodesic distance between p and q. Let $p \in U_1, q \in U_2$, such that $U_1 \cup U_2 \subseteq U$, and choose $d' = \inf\{d(r, s) \mid r \in U_1, s \in U_2\}$.

As $p \neq q$ are limit points of $\{\gamma(t) : t \in \mathbb{R}_+\}$, the flow line traverses atleast distance d' with bounded velocity $\operatorname{grad} f$, thus, if $\gamma(t) \notin U$, then $\exists\, \delta > 0$, such that $\gamma(t - \delta, t + \delta) \subseteq X \setminus U$ (i.e., in particular, $\gamma(t)$ stays outside U, for infinite time).

As $|\operatorname{grad} f|$ is bounded below on $X \setminus U$, for any flow segment $\alpha : [0, 1] \to X \setminus U$, there is a $t \in [0, 1]$ such that $f(\alpha(1)) - f(\alpha(0)) = (f \circ \alpha)'(t) = f'(\alpha(t)) \cdot [-\operatorname{grad} f(\alpha(t))] = -|f'(\alpha(t))|^2 \leq b$ for some constant $b > 0$.

This implies that if t_n are such that $\gamma(t_n) \notin U$, $\lim_{t\to\infty} f(\gamma(t_n)) = -\infty$, a contradiction, as X is compact. □

vspace{1cm}

Problem 2. Suppose γ_n is a sequence of flow lines in $\mathcal{M}(p, q)$. Then there exists a subsequence, a limit k -broken flow line $p \xrightarrow{\bar{\gamma_0}} r_1 \xrightarrow{\bar{\gamma_1}} r_2 \to \cdots \xrightarrow{\bar{\gamma_k}} q$ and sequences $s_{n,0} < s_{n,1} < \cdots < s_{n,k}$ such that $\gamma_n(s_{n,i} + \bullet) \to \bar{\gamma_i}$ uniformly on compact sets.

Solution. Let U be a small neighborhood of p, not containing critical points other than p. Fix $\varepsilon > 0$ such that $B(p, \varepsilon) \subseteq U$ (where this sphere is defined with respect to the Riemannian metric on X). As $\lim_{t\to-\infty} \gamma_n(t) = p$, for every n, there exists a time $s_{n,0}$ such that $\gamma_n(s_{n,0}) \in \partial B(p, \varepsilon)$ and $\gamma_n(t) \in B(p, \varepsilon)\ \forall\, t < s_{n,0}$.

Consider the smooth flow lines $\gamma_n(s_{n,0} + \bullet)$ on the compact set X, Arzela-Ascoli applies and therefore, there is a subsequence converging uniformly on compact sets to a curve $\bar{\gamma_0}$ in X. $\bar{\gamma_0}$ is a *flow* line, because $\operatorname{grad} f$ is smooth and hence (passing to a subsequence without change of notation) as $\gamma_n \to \bar{\gamma_0}$ uniformly on compact sets, $\gamma_n' = -\operatorname{grad} f(\gamma_n)$ converge, in particular, to $\bar{\gamma_0}'$.

We now show that $\bar{\gamma_0}$ is a non-constant flow line and that $\lim_{t\to-\infty} \bar{\gamma_0}(t) = p$. As $\gamma_n(s_{n,0}) \in \partial B(p, \varepsilon)\ \forall\ n$, and $\gamma_n(s_{n,0}) \to \bar{\gamma_0}(0)$, it follows that $\bar{\gamma_0}(0) \in \partial B(p, \varepsilon)$, thus, $\bar{\gamma_0}$ is a non-constant flow line. Further, as $\gamma_n(s_{n,0}) \in B(p, \varepsilon)\ \forall\ t < s_{n,0}\ \forall\ n$, it follows that $\bar{\gamma_0}(t) \in B(p, \varepsilon)\ \forall\ t < 0$. As $t \to -\infty$, $\bar{\gamma_0}(t)$ must converge to a critical point, which thus has to be p.

Let $r_1 = \lim_{t\to\infty} \bar{\gamma_0}(t)$. If $r_1 = q$, we are done. Otherwise, we use the following arguments.

We first procure a ball around r_1 such that any flow line in $W^u(q)$ entering a smaller ball, must leave the original ball at a height lower than that of r_1.

Consider a *small* neighborhood around r_1, and let $B(r_1, \varepsilon)$ be in this neighborhood for some fixed ε. Consider the concentric regions $U_i = \{p \in X \mid i\varepsilon/3 \leq d(p, r_1) \leq (i + 1)\varepsilon/3\}$ for $i = 0, 1, 2$ constituting this ball. As U_1 is compact, $|\operatorname{grad} f|$ is bounded above and below on U_1 by constants $M, m > 0$ respectively. We claim that any flow line on entering

U_1 must spend at least a fixed amount of time in U_1 before leaving it. This would help us to show that every flow line entering a small enough ball around r_1 loses height by at least a fixed amount, which leads us to our final goal, which is the existence of a $\delta > 0$, such that any flow line entering $B(r_1, \delta)$ leaves $B(r_1, \varepsilon)$ at a height lower than that of r_1[1].

More precisely, if γ is a flow line, such that $\gamma(t) \in \partial B(r_1, \varepsilon/3)$ and $\gamma(t + \Delta) \in \partial B(r_1, 2\varepsilon/3)$, then we must have $\Delta \geq \inf\{d(x, y) \mid x \in \partial U_0, y \in \partial U_1 \setminus \partial U_0\}/\sup\{|\operatorname{grad} f(u)| \mid u \in U_1\} = \varepsilon/3M$. Moreover,

$$f(\gamma(t + \Delta)) - f(\gamma(t)) = (f \circ \gamma)'(t')\Delta = f'(\gamma(t'))\gamma'(t')\Delta = -|f'(\gamma(t'))|^2\Delta \leq -m^2\varepsilon/3M =: -h$$

for some $t' \in (t, t + \delta)$.

Now we must choose δ judiciously, so that any flow line of $W^u(q)$ entering $B(r_1, \delta)$ must leave $B(r_1, \varepsilon)$ (which it can do only by traversing across U_1) at a height lower than $f(r_1)$. That is, we want the fall in height h to be at least more than δ i.e., $\delta < h = m^2\varepsilon^2/3M$.

As $\lim_{t\to\infty} \bar{\gamma_0}(t) = r_1$ and $\gamma_n(s_{n,0} + \bullet) \to \bar{\gamma_0}$ uniformly on compact sets, let t be such that $\bar{\gamma_0}(t) \in B(r_1, \delta/3)$ and N be such that for all $n \geq N$, $\gamma_n(s_{n,0} + t) \in B(\bar{\gamma_0}(t), \delta/3) \subseteq B(r_1, \delta)$. Thus, re-labelling γ_n as γ_{n+N}, we see that all flow lines γ_n pass through the $B(r_1, \delta)$ ball. So they must all leave $B(r_1, \varepsilon)$ at a height lower than $f(r_1)$.

Let $s_{n,1} > s_{n,0} + t$ be the *least* time such that $\gamma_n(s_{n,1}) \in \partial B(r_1, \varepsilon)$. Consider the new sequence of flow lines $\gamma_n(s_{n,1} + \bullet)$. As before, by Arzela-Ascoli, it has a convergent subsequence, which without change of notation, we again denote by γ_n. Denote the limit by $\bar{\gamma_1}$. It is a flow line, by convergence of γ_n' to $\bar{\gamma_1}'$, which then has to satisfy the flow equation.

We now claim that $\bar{\gamma_1}$ is a non-constant flow line satisfying $\lim_{t\to-\infty} \bar{\gamma_1}(t) = r_1$. It is non-constant since $\lim_{n\to\infty} \gamma_n(s_{n,1}) = \bar{\gamma_1}(0) \in \partial B(r_1, \varepsilon)$, which comprises regular points.

Let t_0 be such that $\bar{\gamma_0}(t_0) \in B(r_1, 2\varepsilon/3)$ *above* r_1, and let d_n be a positive increasing sequence such that $\lim_{n\to\infty} d_n = \infty$. Then for all n, there exist N_n, such that $\forall\ m \geq N_n, \sup_{t\in[t_0,t_0+d_n]} d(\gamma_m(s_{m,0} + t), \bar{\gamma_0}(t)) < \varepsilon/3$, in particular, $\gamma_m(s_{m,0} + t) \in B(r_1, \varepsilon)\ \forall\ t \in [t_0, t_0 + d_n]$. That is, γ_m spend increasingly larger amounts of time in $B(r_1, \varepsilon)$.

In particular, $\gamma_m(s_{m,0}+\overline{t_0 + d_n}) \in B(\bar{\gamma_0}(t_0+d_n), \varepsilon/3)$ does not belong to $\partial B(r_1, \varepsilon)$. Thus, $s_{m,1} > s_{m,0}+t_0+d_n\ \forall\ m \geq N_n$ as $s_{m,1}$ is the least time after $s_{n,0} + t_0$[2] at which γ_m leaves $B(r_1, \varepsilon)$. In particular, for $t \in [-d_n, 0]$, $\gamma_m(s_{m,1} + t) \in B(r_1, \varepsilon)\ \forall\ m \geq N_n$. Taking limits as $m \to \infty$, this gives us that $\bar{\gamma_1}(t) = \lim_{m\to\infty} \gamma_m(s_{m,1} + t) \in \bar{B}(r_1, \varepsilon)$ for $t \in [-d_n, 0]$. But note that this occurs for all n, and hence taking $n \to \infty$, we get $\bar{\gamma_1}(t) \in \bar{B}(r_1, \varepsilon)$ for $t < 0$. This gives us that the only critical point that $\lim_{t\to-\infty} \bar{\gamma_1}(t)$ could be, has to be r_1.

We now repeat the argument with $\bar{\gamma_1}$ and continue till q is reached, which it must, as X is compact and there are finitely many critical points.

[1] Note that this is easier than arguing existence of ε such that any flow line entering $B(r_1, \varepsilon)$ leaves the *same ball* at a height lower than $f(r_1)$; but this suffices for our purpose.

[2] Note that this is *not* the definition of $s_{n,1}$ but follows as an easy consequence.