Floer theory : Homework 2

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Problem 1. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. If $u \in W^{k,p}(\Omega)$ with not necessarily compact support (in $\mathbb{R}^n$), then given any $\varepsilon > 0$, there is a smooth function $u \in \mathscr{C}^\infty(\bar{\Omega})$ such that $\|u' - u\|_{W^{k,p}(\Omega)} < \varepsilon$ with $\mathrm{supp}(u')$ compact in $\mathbb{R}^n$.

Solution. Following the strategy outlined in class, we first do this problem when u has compact support in Ω. The idea is to use mollifiers (smoothing operators) to approximate u by smooth functions.

Consider $u \in W^{k,p}(\Omega)$ with compact support. Let $\rho : \mathbb{R}^n \to \mathbb{R}$ smooth such that $\mathrm{supp}(\rho) \in B(0,1)$ and $\int_{\mathbb{R}^n} \rho = 1$. For $\delta > 0$ small, define $\rho_\delta = \rho(\delta^{-1}x)/\delta^n$. This has the nice property that $\int_{\mathbb{R}^n} \rho_\delta = 1$. Now we convolve ρ_δ with u,

$$u_\delta := u * \rho_\delta : x \mapsto \int_{\mathbb{R}^n} u(x-y)\rho_\delta(y)\ \mathrm{d}y.$$

We first note that u_δ is smooth. Indeed, if ∂_ν denotes partial derivative with respect to a coordinate function ν, then

$$\partial_\nu u_\delta = \partial_\nu(u * \rho_\delta) = u * \partial_\nu \rho_\delta = \partial_\nu u * \rho_\delta$$

following the definition of weak derivative (integration by parts). Thus, one can keep differentiating u_δ by differentiating ρ_δ. In particular, since u_δ has compact support, $u_\delta \in W^{k,p}(\mathbb{R}^n)$.

Now, we show that if $f \in \mathscr{C}^0_{\mathrm{loc}}(\Omega)$, then $f_\delta \to f$ in $\mathscr{C}^0_{\mathrm{loc}}$. Indeed,

$$f(x) - f_\delta(x) = f(x)\int_{\mathbb{R}^n} \rho_\delta(y)\ \mathrm{d}y - \int_{\mathbb{R}^n} f(x-y)\rho_\delta(y)\ \mathrm{d}y = \int_{\mathbb{R}^n} (f(x) - f(x-y))\rho_\delta(y)\ \mathrm{d}y$$

and as f is continuous on its compact support, it is uniformly continuous, thus given $\varepsilon > 0$, there is a $\delta > 0$, such that for any x and for all $|y| < \delta$, $|f(x) - f(x-y)| < \varepsilon$. Thus we get,

$$|f(x) - f_\delta(x)| \leq \varepsilon \int_{\mathbb{R}^n} |p_\delta(y)|\ \mathrm{d}y = \varepsilon.$$

Using the fact that $\mathscr{C}^0_{\mathrm{loc}}(\Omega)$ is dense in $\mathscr{L}^p(\Omega)$ and an analogue of the above series of inequalities with $\mathscr{L}^p$ norms (noting $\|\rho_\delta\|_{\mathscr{L}^p(\mathbb{R}^n)} < \infty$), one concludes that if $f \in \mathscr{L}^p(\Omega)$, then $f_\delta \to f$ in $\mathscr{L}^p$. Alternatively, one may use the Young's inequality to show this.

Thus if $u \in W^{k,p}(\Omega)$, we can do the above for each of the weak derivatives of u of degree $\leq k$, thereby, giving that $u_\delta \to u$ in $W^{k,p}(\Omega)$.

Now we consider the case when u does not have compact support in Ω. Our approach shall be to first approximate u with functions in $W^{k,p}_{\mathrm{loc}}(\Omega)$, and then to approximate the latter with compactly supported smooth functions (which we showed above). This hints that we multiply u with bump functions to make it compactly supported, and hope that the resultant functions approximate u in $\mathscr{L}^p$ (and hopefully $W^{k,p}$). This is not always possible, if u has fast growth near the boundary. Since $u \in W^{k,p}(\Omega)$ and Ω has smooth boundary, we shall control the behavior of u in a band near the boundary of Ω.

For $A \subseteq \mathbb{R}^n$, define $d(x, A) := \inf\{|x-a| \mid a \in A\}$ and let $\Omega_\delta := \{x \in \Omega \mid d(x, \Omega^c) \leq \delta\}$. By smoothness (and compactness) of $\partial\Omega$, there is a δ such that $\Omega_\delta \simeq (0,\delta] \times \partial\Omega$. Our further choices of δ will be chosen such that this is satisfied. Note that $\bar{\Omega}_\delta$ is compact since Ω is bounded. We claim that given $\varepsilon > 0$, there is a δ such that $\int_{\Omega_\delta} |u|^p < \varepsilon$. This is proved by assuming the contrary and using the same trick as proving that integrable functions f (that is, $\int_{\mathbb{R}} f < \infty$) satisfy $\int_U f \to 0$ as U gets smaller. Namely, pick a $\tilde{u} \in \mathscr{C}^0_{\mathrm{loc}}(\Omega)$ such that $\int_\Omega |u - \tilde{u}|^p < \varepsilon/2$. Note that $\tilde{u}$ is uniformly continuous on its support, and consider for $f \in \mathscr{L}^p(\Omega_\delta)$, $\int f$ as function on $(0,\delta]$ using the identification $\int_0^\delta f := \int_{\Omega_\delta} f$. In this notation, note that there is a K, so that $\int_\delta^{\delta+T} |\tilde{u}|^p \leq KT^p$. Letting $T = 1/2^n$, we get a contradiction if we assume $\int_{\Omega_\delta} |\tilde{u}|^p \geq m\ \forall\ \delta$ for some fixed m (partition $(0,\delta]$ with sets of the form $[1/2^{n+1}, 1/2^n]$ and note that the integral $\int |\tilde{u}|^p$ can differ between successive intervals by an amount $K/2^{np}$, which is absolutely summable. But then $\int_{1/2^n}^\delta |\tilde{u}|^p$ differs from nm by a bounded quantity. As $n \to \infty$, this gives us a contradiction).

Once the above is done, choose a smooth bump function $\rho : \bar{\Omega} \to [0,1]$ so that $\rho|_{\Omega \setminus \Omega_\delta} \equiv 1$ and $\rho|_{\partial\Omega} \equiv 0$. Then, we see that $\rho u \in W^{k,p}_{\mathrm{loc}}(\Omega)$, and moreover,

$$\int_\Omega |u - \rho u|^p = \int_{\Omega_\delta} |u|^p |1-\rho|^p \leq \int_{\Omega_\delta} |u|^p < \varepsilon.$$

Likewise, we have inequalities, for the weak derivatives of u (that is, we choose δ so that $\int_{\Omega_\delta} |Du|^p < \varepsilon$ for all weak derivatives Du of u of degree $\leq k$). Thus, we have shown that we can approximate to an arbitrary extent u by a compactly supported ρu. Combining with the result before, we conclude that we can approximate u to an arbitrary extent by a compactly supported smooth function.

Problem 2. Let $\Omega_0, \Omega_1 \subseteq \mathbb{R}^n$ be bounded domains with $\partial\Omega_0, \partial\Omega_1$ smooth. Further let $\phi : \bar{\Omega_0} \to \bar{\Omega_1}$ be a $\mathscr{C}^l$ diffeomorphism. Then $\phi^* : W^{k,p}(\Omega_1) \to W^{k,p}(\Omega_0)$ given by $f \mapsto f \circ \phi$ is bounded if $l \geq k$ and $\|\phi\|_{\mathscr{C}^l}, \|\phi^{-1}\|_{\mathscr{C}^l} < \infty$.

Solution. Let K denote $\max\{\|\phi\|_{\mathscr{C}^l}, \|\phi^{-1}\|_{\mathscr{C}^l}\}$, and note that if D is a derivative operator of degree $\leq l$, then $\|D\phi\| \leq K, \|D\phi^{-1}\| \leq K$.

So that we can freely use the chain rule of strong derivatives, we shall prove the result for $\phi^* : \mathscr{C}^\infty(\bar{\Omega_1}) \to \mathscr{C}^\infty(\bar{\Omega_0})$, and as the latter spaces are dense in the respective Sobolev spaces (by Problem 1), we shall be done, by extension of bounded linear operators defined on dense subspaces.

By taking a more abstract viewpoint (thinking of derivatives as bundle maps between tangent bundles), we shall present a short formulation of chain rule. Let D denote the derivative operator.

Note that $D(f \circ \phi) = Df(\phi) \circ D\phi = (Df)(\phi, D\phi)$.

Taking derivatives again, $D^2(f \circ \phi) = D((Df)(\phi, D\phi)) = (D^2 f)(\phi, D\phi) \circ (D\phi, D^2\phi) = (D^2 f)(\phi, D\phi, D\phi, D^2\phi)$.

Similarly, $D^r(f \circ \phi)$ will turn out to be $(D^r f)[v(\phi)]$ where $v(\phi)$ is a vector with each coordinate being the derivative of some order of ϕ. Let Φ_r denote $v(\phi)$. Since ϕ is a $\mathscr{C}^l$ diffeomorphism and $r \leq k \leq l$, we note that $|\Phi_r|_{\mathscr{L}^p} \leq K, |\Phi_r^{-1}|_{\mathscr{L}^p} \leq K$. Thus we look at Φ_r as a $\mathscr{C}^{l-r}$ diffeomorphism from $T^r(\bar{\Omega_0})$ to $T^r(\bar{\Omega_1})$ where T^r denotes tangent bundle of the tangent bundle of the ... (r times) of the respective $\Omega_i, i = 1, 2$.

Now note that $\int_{\Omega_0} |D^r(f \circ \phi)|^p = \int_{\Omega_0} |D^r f \circ \Phi_r|^p \leq K \int_{\Omega_0} |D^r f \circ \Phi_r|^p \cdot |\operatorname{Jac}(\Phi_r)| = K \int_{T^r\Omega_1} |D^r f|^p$ by the change of variables formula, where Jac denotes the Jacobian. And we know that the last quantity is bounded by $K\|f\|^p_{W^{k,p}(\Omega_1)}$. Varying r from 1 to k, we get that ϕ^* is a bounded linear operator.