Floer theory : Homework 3

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Problem 1. Let γ_n be flow lines such that $\gamma_n \to \gamma$ in $\mathscr{C}^\infty_{\text{loc}}(\mathbb{R})$. Suppose $\lim_{t\to\infty} \gamma_n(t) = \lim_{t\to\infty} \gamma(t) = y$. Then for any $\varepsilon > 0$, there are N, T, such that $\gamma_n(t) \in B_\varepsilon(y)\ \forall\ t \geq T, n \geq N$.

Solution. We shall only need that $\gamma_n \to \gamma$ pointwise, and we shall utilize a strategy similar to the solution of HW 1, Problem 1. Let $\varepsilon > 0$ be given.

Consider a neighborhood U around y, and let $B(y, \varepsilon)$ be in this neighborhood. Consider the concentric regions $U_i = \{p \in X \mid i\varepsilon/3 \leq d(p, y) \leq (i+1)\varepsilon/3\}$ for $i = 0, 1, 2$ constituting this ball. As U_1 is compact, $|\operatorname{grad}(f)|$ is bounded above and below on U_1 by constants $M, m > 0$ respectively. We claim that any flow line on entering U_1 must spend a fixed amount of time in U_1 before leaving it. This would help us to show that every flow line entering a small enough ball around y loses height by at least a fixed amount if it were to leave $B(y, \varepsilon)$, which would for flow lines ending at y, show the existence of δ so that flow lines entering a small enough ball $B(y, \delta)$ will never leave $B(y, \varepsilon)$.

More precisely, if γ is a flow line, such that $\gamma(t) \in \partial B(y, \varepsilon/3)$ and $\gamma(t + \Delta) \in \partial B(y, 2\varepsilon/3)$, then we must have $\Delta \geq \inf\{d(x, y) \mid x \in \partial U_0, y \in \partial U_1 \setminus \partial U_0\}/\sup\{|\operatorname{grad}(f(u))| \mid u \in U_1\} = \varepsilon/3M$. Moreover,

$$f(\gamma(t+\Delta)) - f(\gamma(t)) = (f \circ \gamma)'(t')\Delta = f'(\gamma'(t))\gamma'(t')\Delta = -|f'(\gamma(t'))|^2\Delta \leq -m^2\varepsilon/3M =: -h$$

for some $t' \in (t, t + \Delta)$.

Now we must choose δ judiciously, so that any flow line entering $B(y, \delta)$ can only leave $B(y, \varepsilon)$ (which it can do only by traversing across U_1) at a height lower than $f(y)$. That is, we want the fall in height h to be atleast more than δ i.e., $\delta < h = m^2\varepsilon^2/3M$. So for flow lines ending at y, we see that they cannot traverse across U_1 once they enter $B(y, \delta)$, that is, they cannot leave $B(y, \varepsilon)$.

Now let T be such that $\gamma(t) \in B(y, \delta/2)\ \forall\ t \geq T$. And let N be such that $\gamma_n(T) \in B(\gamma(T), \delta/2)\ \forall\ n \geq N$. Then we have $\gamma_n(T) \in B(y, \delta)\ \forall\ n \geq N$. Since γ_n has entered $B(y, \delta) \subseteq B(y, \varepsilon)$ at time T, from the above it follows that it can't leave $B(y, \varepsilon)$ and we are done.