Floer theory : Homework 4

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Problem 1. Let f be a Morse function on a closed manifold M. Suppose $\xi : \mathbb{R} \to \mathbb{R}^n$ is a (trivialized) vector field path along a Morse trajectory γ (that is, a section of $\gamma^* TM \simeq \mathbb{R} \times \mathbb{R}^n$). Consider the linear operator $A : \mathrm{Hom}(\mathbb{R}, \mathbb{R}^n) \to \mathrm{Hom}(\mathbb{R}, \mathbb{R}^n)$, given by

$$\xi \mapsto \nabla_\xi \operatorname{grad}(f), \text{ or more precisely, } (t \to \xi(t)) \mapsto \big(t \to (\nabla_{\xi(t)} \operatorname{grad}(f))(t)\big)$$

Since $\nabla_\bullet$ is tensorial, we can think of A as the time dependent operator $A(t) : T_{\gamma(t)}M \to T_{\gamma(t)}M$, given by $v \mapsto (\nabla_v \operatorname{grad} f)(\gamma(t))$. Show that $\lim_{t\to\pm\infty} A(t)$ are symmetric and non-degenerate.

Solution. We show the result for $t \to \infty$ and the proof is similar for $t \to -\infty$.

We use $\langle \cdot, \cdot \rangle$ to denote the Riemannian metric and $\tilde{X}$ to denote for a tangent vector $X \in T_pM$, its extension to a local vector field near p with $\tilde{X}_p = X$. We shall show in fact, that $A(t) : T_{\gamma(t)}M \to T_{\gamma(t)}M$ is symmetric for all t. For $v, w \in T_{\gamma(t)}M$, (all computations occurring at $\gamma(t)$)

$$\begin{aligned} \langle A(t)v, w \rangle &= \langle (\nabla_v \operatorname{grad}(f)), w \rangle \\ &= v \langle \operatorname{grad}(f), \tilde{w} \rangle - \langle \operatorname{grad}(f), \nabla_v \tilde{w} \rangle \\ &= v(\tilde{w}(f)) - \mathrm{d}f(\nabla_v \tilde{w}) \end{aligned}$$

In particular, since $\langle A(t)v, w \rangle$ is $\mathscr{C}^\infty(M)$ -linear in both v and w and depends locally on v, w, it suffices to prove symmetry taking $\tilde{v} = \partial/\partial x_i, \tilde{w} = \partial/\partial x_j$. In particular, $[\tilde{v}, \tilde{w}]$ would then by 0, so

$$\langle A(t)v, w \rangle = v(\tilde{w}(f)) - \mathrm{d}f(\nabla_v \tilde{w}) = w(\tilde{v}(f)) - \mathrm{d}f(\nabla_w \tilde{v}) = \langle v, A(t)w \rangle.$$

Thus, $A(t)$ is symmetric for all t. Now, to show non-degeneracy at the limit, take $v, w \in T_{\gamma(\infty)}$. Consider the expression $\langle A(t)\tilde{v}_{\gamma(t)}, \tilde{w}_{\gamma(t)} \rangle = \tilde{v}_{\gamma(t)}(\tilde{w}(f)) - \mathrm{d}f_{\gamma(t)}(\nabla_{\tilde{v}_{\gamma(t)}} \tilde{w})$ as a function of t. As $f, \tilde{v}, \tilde{w}$ are smooth on M, these are smooth in t; moreover, $\lim_{t\to\infty} A(t)$ exists as A as a family of maps $\xi \to \nabla_\xi \operatorname{grad}(f)$ is bounded. So (using: if $A_t \to A$ in the operator norm, $v_t \to v$ then $A_t v_t \to Av$) taking limits as $t \to \infty$, we get:

$$\left\langle \left(\lim_{t\to\infty} A(t) \right) v, w \right\rangle = v(\tilde{w}(f)) \quad \text{as} \quad \lim_{t\to\infty} \mathrm{d}f_{\gamma(t)} = 0.$$

But the above limit is the Hessian at $\gamma(\infty)$, which is a critical point. Since f is Morse, this Hessian is non-degenerate.