Floer theory : Homework 5

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Problem 1. Prove that for any $l \geq 2$, the set of $\mathscr{C}^l$ Morse functions on a compact manifold is dense in $\mathscr{C}^l$.

Solution. The idea is: for a given $\mathscr{C}^l$ function f, to construct a Morse function $\widetilde{f}$ close to it, by in fact, perturbing f to patch any degenerate critical points it might have. For the correction factors, a natural choice of functions is local coordinate functions. Thus, we take a collection of coordinate functions and add to f locally, so that the Hessian is always nondegenerate. We should also ensure that no new critical points are created, for which we'll use an application of Sard's theorem.

For convenience of notation, let $\{(U_\alpha, \varphi_\alpha)\}$ be a finite atlas covering the compact manifold M, by N charts. And let $(x_\alpha^1, \dots, x_\alpha^n)$ be coordinate functions on U_α, extended to smooth functions on all of M (by shrinking U_α and using bump functions to extend x_α^i by 0 on all of M).

Note that making sure that $\mathrm{d}f$ as a section of T^*M is transverse to the zero section is the same as ensuring that the Hessian is non-degenerate. Consider the perturbed differential of f, as a section of T^*M, via the map $\Phi : \mathbb{R}^{Nn} \times M \to T^*M$ (to be thought of as the family $\Phi : \mathbb{R}^{Nn} \to \Gamma(M, T^*M)$) given by

$$\Phi : (\lambda_{\alpha,i}, x) \mapsto \mathrm{d}f(x) + \sum_{\alpha, i=0,\dots,n} \lambda_{\alpha,i}\, \mathrm{d}x_\alpha^i(x).$$

We first claim that Φ is transverse to the zero section of T^*M. For any point contained in U_α, if we set $\lambda_{\beta,i} = 0$ for all $\beta \neq \alpha \;\forall\, i$, we get

$$\mathrm{d}f + \sum_{i=1}^{n} \lambda_{\alpha,i}\, \mathrm{d}x^i.$$

By varying the constants $\lambda_{\alpha,i}$ above, we can span all of T_x^*M, showing that Φ is a submersion.

To avoid the problem around creation of new critical points, we use an abstract existence argument. The following is a consequence of Sard's theorem:

Lemma 0.1. Let $F : X \times Y \to Z$ be a smooth map, for smooth manifolds X, Y, Z. Suppose $W \subseteq M$ is a submanifold such that $F \pitchfork W$. Then the set $\{x \in X \mid F(x, \cdot) : Y \to Z \text{ with } F(x,\cdot) \pitchfork Q\}$ is dense in X.

Proof. We shall abbreviate $F(x,\cdot)$ as f_x (to avoid confusing $(\mathrm{d}F)_x$ with $\mathrm{d}f_x$). We know that $F^{-1}(W) \subseteq X$ is a submanifold as $F \pitchfork W$. Let $\pi : F^{-1}(W) \to X$ be the restriction of the first projection. We claim that if x is a regular value of π, then $f_x \pitchfork W$. The lemma then follows from Sard's theorem. The proof is a careful bookkeeping of tangent vectors.

Let y be such that $z := F(x,y) \in W$, let $\widetilde{v} \in T_z Z$. Since $F \pitchfork W$, we can write $\widetilde{v} = w + \mathrm{d}F_{(x,y)} v$ for some $w \in T_z W$ and some $v \in T_{(x,y)}(X \times Y)$. Suppose also that $v = v_1 + v_2$ for $v_1 \in T_x X, v_2 \in T_y Y$.

Suppose that x is a regular value of π, that is, $\mathrm{d}\pi_{(x,y)}(T_{(x,y)}(F^{-1}W)) = T_x X$. Note that, from the transversality result (that $F^{-1}W$ is a smooth manifold) we also have $T_{(x,y)}F^{-1}W = (\mathrm{d}F_{(x,y)})^{-1}(T_z W)$, thus there is a vector $\widetilde{v_1} \in T_{(x,y)}(F^{-1}W) \subseteq T_{(x,y)}(X \times Y)$ such that $\mathrm{d}\pi_{(x,y)}(\widetilde{v_1}) = v_1$ and $\mathrm{d}F_{(x,y)}(\widetilde{v_1}) \in T_z W$. Since $\mathrm{d}\pi$ is just the first projection, we have $\widetilde{v_1} - v_1 \in T_y Y$ (or more precisely, $\widetilde{v_1} - v_1 \in T_{(x,y)}(\{x\} \times Y)$). In the xth slice of $X \times Y$,

$$(\mathrm{d}F|_{T_y(\{x\}\times Y)})_{(x,y)} = \mathrm{d}(f_x)_y,$$

and so finally, we can write $\widetilde{v} = w + \mathrm{d}F_{(x,y)}(v) = (w + \mathrm{d}F_{(x,y)}(\widetilde{v_1})) + \mathrm{d}(f_x)_y(v_2 + \overline{v_1 - \widetilde{v_1}})$ (adding and subtracting $\mathrm{d}F_{(x,y)}\widetilde{v_1}$). Thus, $f_x \pitchfork W$. □

Intuitively, since F is transverse to W, critical values of F in W give rise to critical "points" in X, which have measure 0, since they arise as the critical values of the projection restricted to $F^{-1}W$. Applying the above parametric transversality argument to Φ, the set of parameters $\lambda_{\alpha,i}$ such that Φ_λ is transverse to the zero section is dense. Hence we can find λ arbitrarily small such that

$$f + \sum \lambda_{\alpha,i} x_\alpha^i$$

is a Morse function. Since λ can be arbitrarily small, this means that one can approach f arbitrarily in $\mathscr{C}^l$.