On J-holomorphic Curves In Symplectic Topology

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To my advisor,
for her incredible kindness and patience

Abstract

A J-holomorphic curve is a (j,J)-holomorphic map $u : \Sigma \to M$ from a Riemann surface (Σ, j) to an almost complex manifold (M,J). When we consider an almost complex structure on a symplectic manifold with compatibility constraints, it is natural to consider J-holomorphic curves in it and consider the space of such curves. These arise as a smooth manifold, which becomes compact in special cases modulo reparametrizations. Gromov used clever embedding arguments and a monotonicity theorem for minimal surfaces to conclude that symplectic embeddings cannot squeeze area. Moreover, when the moduli spaces are compact, one has a robust intersection theory on them and in this setting, the so called Gromov-Witten invariants offer symplectic invariants. In this thesis, we study the basic theory of J-holomorphic curves, and in particular, their moduli space using transversality arguments and elliptic regularity. We also study compactness of this space in some special conditions and finally use this to give a proof of the non-squeezing theorem.

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Contents

Introduction

The J-holomorphic technology introduced in Gromov's 1985 paper for the proof of his famous non-squeezing theorem has been very influential in the modern development of symplectic topology. Over the years, it has had wide applications in important areas like algebraic geometry and mathematical physics. The power of this machinery makes it an attractive tool for research, and thus it is only wise to understand the technical aspects and analytic subtleties underlying the theory. In this thesis, we attempt to give an account of the foundational aspects of J-holomorphic curves, accessible to graduate students, with a focus on motivation and rigorous details for the analysis.

Notation. We denote by $B^n(x,r)$ or $B^n_r(x)$, the open ball in $\mathbb{R}^n$ of radius r centered at x. Whenever one of the parameters n, x, r is omitted, their meaning is derived from context or from standard conventions. For instance, when x is omitted, we mean $x = \vec{0}$, that is, $B^n(r) := B^n(\vec{0}, r)$. We shall use ω_{std} to denote the standard symplectic form on the Euclidean space (of any given dimension), as well as on tori defined by $T^n := \mathbb{R}^n/\lambda\mathbb{Z}^n$ for $\lambda > 0$. Note that ω_{std} descends to a symplectic form on T^n since it is invariant under the action of $\mathbb{Z}^n$ on $\mathbb{R}^n$ by translations.

For $\Omega \subseteq \mathbb{R}^n$ open, denote by $\mathscr{C}^\infty(\overline{\Omega})$ the space of restrictions of smooth functions on $\mathbb{R}^n$ to $\overline{\Omega}$ and by $\mathscr{C}_0^\infty(\Omega)$ the space of smooth compactly supported functions on Ω.

In the following, all manifolds are smooth and without boundary, and all Riemann surfaces are closed unless otherwise specified.

Chapter 1

Preliminaries

In this chapter, we introduce the relevant language of symplectic topology and give some basic properties of J-holomorphic curves.

1.1 Almost complex structures

Past study of elliptic PDEs (which have wide applications throughout mathematics) has shown that a lot of the rich behavior of complex analytic functions comes about because their defining equation, the Cauchy Riemann equation, is elliptic. In particular, if the equation is slightly generalized while preserving ellipticity, it is only natural to expect many of its solutions' properties to be preserved. In this context, we introduce almost complex structures as a first generalization to the CR equation.

1.1.1 Linear structures

Given a real vector space V, an *almost complex structure* J on it is a linear automorphism satisfying $J^2 = -\mathbb{1}$. One can show in various ways, that in such a case, dimension of V has to be even. Conversely, any even dimensional real vector space has almost complex structures on it. Again, using elementary linear algebra, it can be shown that all such structures are conjugate to the following ("standard") structure

$$J_{\text{std}} = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

J is to be thought of as *multiplication by* i, although clearly, there is no scalar i over the real numbers. More precisely, V can be treated as a complex vector space, defining $i \cdot v = Jv$ for a vector v, and hence we can talk of an $\mathbb{R}$-linear operator on V to be complex linear or complex antilinear.

1.1.2 On vector bundles

The above notions can be generalized to vector bundles over a topological space, and can be further restricted analytically, for instance, by requiring J_x to vary with x while being smooth, $\mathscr{C}^k$ or $\mathcal{W}^{k,p}$ (defined in Section B.1).

More precisely, an *almost complex structure on* $E \to M$ (a real vector bundle of even rank) is a continuously varying family of complex structures on it's fibres, such that as a bundle automorphism, $J^2 = -\mathbb{1}$. In such a case, since each J_x induces a complex structure on the fiber over x, E acquires the structure of a complex vector bundle.

In our subsequent discussion, we shall assume that $E \to M$ is a smooth vector bundle and that J is smooth (unless otherwise specified).

1.1.3 On manifolds

For a smooth even dimensional manifold M, by an *almost complex structure on it*, we mean one such on its tangent bundle. M is then called an *almost complex manifold*.

The use of the word "almost" can now be justified.

Suppose M is a complex manifold of complex dimension n. Any choice of holomorphic local coordinates on a subset $U \subseteq M$ identifies the tangent spaces T_pU with $\mathbb{C}^n$. If we use this identification to assign the standard complex structure i to each tangent space T_pU, then the fact that the transition maps are holomorphic (in particular, that their derivatives are $\mathbb{C}$-linear) implies that this assignment doesn't depend on the choice of coordinates. Thus M has a natural almost complex structure J that looks like the standard complex structure in any holomorphic chart.

The above shows that almost complex manifolds resemble complex manifolds at the level of their linearizations (tangent spaces), and brings up the question of whether all almost complex manifolds admit complex structures. The following section answers this in the negative.

1.1.4 Integrability

An almost complex structure is said to be *integrable* if it arises from a system of holomorphic coodinate charts – in such a case, we just call J a complex structure on M, that is, M is actually a complex manifold, with J being the induced structure on it's (real) tangent bundle.

The notion of integrability is prevalent in differential geometry. Often, a structure on a manifold is described in terms of a linear structure on the tangent bundle, combined with an integrability condition (like closedness of a certain form, or closure under Lie brackets, etc). In the language of structure groups, integrability is often described in terms of local flatness.

The holomorphic tangent bundle. In the complex category, we would like our tangent bundle to be a holomorphic vector bundle. Given a system $\{(U_\alpha, \varphi_\alpha)\}$ of local holomorphic trivializations, one can defined $T^c M$ as the union of $U_\alpha \times \mathbb{C}^n$, glued by identifying $U_i \cap U_j \times \mathbb{C}^n \subseteq U_i \times \mathbb{C}^n$ and $U_i \cap U_j \times \mathbb{C}^n \subseteq U_j \times \mathbb{C}^n$ via

$$(u, v) \mapsto (u, \varphi_{ij_*}(v))$$

where ϕ_{ij_*} is the holomorphic Jacobian matrix with holomorphic coefficients $\partial \varphi^k_{i,j}/\partial z_\ell(u)$. This gives us the holomorphic tangent bundle, temporarily denoted by $T^c M$. Just like in the smooth case, it can also be defined as the set of $\mathbb{C}$-valued derivations of the $\mathbb{C}$-algebra of holomorphic functions (really *local holomorphic germs*, but we can just say functions because of identity principle), or as the set of jets of order 1 of holomorphic maps from the complex disk to X. The bundle $T^c M$ is generated, in the charts U_α by the elements

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right),$$

under the coordinate system $z_k = x_k + i y_k$ of complex coordinates. The above tangent vectors are naturally elements of the complexified (real) tangent space $T_{\mathbb{C}}M := T_{\mathbb{R}}M \otimes \mathbb{C}$. Under local real trivializations, $T_{\mathbb{R}}U_\alpha \simeq \mathbb{C}^n \times \mathbb{R}^{2n}$, given by the sections $\partial/\partial x_j, \partial/\partial y_j$ of $T_{\mathbb{R}}U_\alpha$. The induced complex structure J_i on $T_{\mathbb{R}}U_\alpha$ sends $\partial/\partial x_j$ to $\partial/\partial y_j$, and gives a map $J_i \otimes \mathbb{1}$ on $T_{\mathbb{C}}M$, denoted again by J_i. Thus the tangent vectors of type $(1,0)$ (i.e., eigenvectors of J_i for eigenvalue i) are generated over $\mathbb{C}$ at each point by

$$\frac{\partial}{\partial x_j} - iJ_i\frac{\partial}{\partial y_j} = 2\frac{\partial}{\partial z_j}.$$

In conclusion, this amounts to saying that for a complex manifold, the sub bundle $T^{1,0}M \subseteq T_{\mathbb{C}}M$, is equal to the holomorphic tangent bundle $T^c M$ (henceforth, we'll only use the notation $T^{1,0}M$).

Similarly, for an almost complex manifold (M, J), $T_{\mathbb{C}}M$ contains a vector sub-bundle, denoted by $T^{1,0}M$ and defined as the bundle of eigenvectors of $J \otimes \mathbb{1}$ for the eigenvalue i. As a real vector bundle, $T^{1,0}M$ is naturally isomorphic to $T_{\mathbb{R}}M$ by taking the real part ($T^{1,0}M$ is generated by $v - iJv$, $v \in T_{\mathbb{R}}M$). Moreover, this identification identifies the operators i on $T^{1,0}M$ and J on $T_{\mathbb{R}}M$. Henceforth, let's assume (M, J) is almost complex.

Complex conjugation acts naturally on $T_{\mathbb{C}}M$, interchanging $T^{1,0}M$ with $T^{0,1}M$, the subbundle of complexified tangent vectors which are eigenvectors of $J \otimes \mathbb{1}$ for the eigenvalue $-i$. Thus, we have a direct sum decomposition into the isotypic components of the $\mathbb{C}^*$-action on $T_{\mathbb{C}}M$ induced by $J \otimes \mathbb{1}$.

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M.$$

Remark 1.1.1. When M is an almost complex manifold, $T^{1,0}M$ does not a priori have the structure of a holomorphic bundle. This is where integrability comes in.

The Lie bracket of (real) vector fields on M extends by $\mathbb{C}$-linearity to the complexified vector fields, i.e., to the differential sections of $T_{\mathbb{C}}M$. If M were an honest complex manifold, we would have that the Lie bracket induces a bracket structure on the holomorphic tangent bundle $T^{1,0}M$. This seems to be the key condition, guaranteeing integrability.

Theorem 1.1.2 (Newlander-Nirenberg). J on M is integrable iff $[T^{1,0}M, T^{1,0}M] \subseteq T^{1,0}M$, or equivalently by conjugation, $[T^{0,1}M, T^{0,1}M] \subseteq T^{0,1}M$.

This theorem is a powerful (and difficult) theorem in analysis. When (M, J) is assumed to be real analytic, the proof is easier, and follows from an analytic version of Frobenius theorem (see [1], Theorem 2.2.4). The general version is considerably more intricate, and can be found in [2], Section E.3.

Let $X - iJX, Y - iJY \in T^{1,0}M$ be vector fields. Then,

$$[X - iJX, Y - iJY] = [X, Y] - [JX, JY] - i([X, JY] + [JX, JY]).$$

The above is in $T^{1,0}M$, that is, of the form $Z - iJZ$, iff $([X, JY] + [JX, JY]) = J([X, Y] - [JX, JY])$. This gives us a useful algebraic way of expressing this criterion for integrability. Denote by $N_J \in \Omega^2(M, TM)$ the *Nijenhuis tensor* of J. It is defined on $X, Y \in T_{\mathbb{R}}M$ as

$$N_J(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]$$

and measures the non-integrability of J. From the above discussion, J is integrable iff N_J vanishes identically. Let us show that the word "tensor" is justified.

Lemma 1.1.3. N_J is tensorial (i.e., it is $\mathscr{C}^\infty(M)$-bilinear) and antisymmetric.

Proof. Antisymmetry of N_J follows from that of $[\cdot, \cdot]$. Using this antisymmetry, it suffices to show that $N_J(fX, Y) = fN_J(X, Y)$. We first recall that

$$\begin{aligned}
[fX, Y](g)|_p &= (fX)|_p(Yg) - Y|_p(fX(g)) \\
&= f(p)X_p(Yg) - Y_p(f)X_p(g) - f(p)Y|_p(Xg) \\
&= f[X, y](g)|_p - Y_p(f)X_p(g)
\end{aligned}$$

so that varying p and g, gives us $[fX, Y] = f[X, Y] - Y(f)X$. Substituting this in the definition of N_J gives our result. □

This gives us that $N_J(X_p, Y_p)$ depends only on the values $X_p, Y_p \in T_pM$, and not on their extensions X, Y (this is the standard argument of pointwise dependence due to $\mathscr{C}^\infty(M)$ linearity). Also worthwhile to note is the evident fact $N_J[X, JX] = 0$. This gives us the following useful lemma.

Lemma 1.1.4. If M is a smooth *surface*, then $N_J \equiv 0$.

Proof. We fix $p \in M$ and compute $N_J(X, Y)|_p$ for two vector fields X and Y. The key point is that $\{X_p, JX_p\}$ (for X nonvanishing at p) furnishes a basis of T_pM. Hence, $N_J(X, Y)|_p = N_J(X_p, \alpha X_p + \beta JX_p) = 0$. □

Using the small steps above, we get something (moderately) profound. Notice that any orientable surface is trivially symplectic (the top form is the symplectic form), and so using a Riemannian metric on M, we get hold of a compatible (defined in the next section) almost complex structure J on M. Thus, using the Newlander-Nirenberg theorem, this J is integrable and thus, (M, J) is a complex manifold. More formally,

Theorem 1.1.5. Every oriented smooth surface admits the structure of a Riemann surface.

1.1.5 Interaction with symplectic structure

Let (M, ω) be a symplectic manifold (that is, M is a smooth manifold with a closed non-degenerate 2-form ω). An almost complex structure on it $J : TM \to TM$ is called ω-tame ("ω tames J") if $v \neq 0 \implies \omega(v, Jv) > 0$, and is called ω-compatible if $\omega(Jv, Jw) = \omega(v, w)$ so that J is a symplectomorphism (of symplectic vector bundles). Geometrically, the former condition means that ω restricts to a positive form on each complex line $L = \mathrm{span}\{v, Jv\}$ in the tangent space T_xM, while the latter makes sure that $J^*\omega = \omega$.

In the ω-compatible case, J determines a Riemannian metric on M, defined by $g(v, w) = \omega(v, Jw)$. The structures g, ω and J are then said to form a *compatible triple*. It can be shown that in a compatible triple, any two structures determine the third (see [1]).

Taming would often be enough for our purposes, i.e., we may not require that the form g defined above be symmetric, but one can always consider a symmetric version:

$$g(v,w) = (\omega(v,Jw) + \omega(w,Jv))/2.$$

Our study will mostly concern spaces of almost complex structures having one or both of the above conditions. An interesting fact is that the space of almost complex structures on M, denoted $\mathcal{J}(M)$, the ones which are ω-tame, denoted $\mathcal{J}_\tau(M,\omega)$, and the ones which are ω-compatible, denoted $\mathcal{J}(M,\omega)$, occur as contractible spaces (even when ω is not closed). A proof can be found in [3], Proposition 4.1.1 or in [4], Homework 9's hint.

1.2 J-holomorphic curves

In the previous section, we have acquired an analogue of i at the level of linearizations of manifolds. In this section, we likewise generalize maps whose derivatives are $\mathbb{C}$-linear (holomorphic maps) to those whose derivatives play well with J. For reasons that will be justified in the next section, we shall restrict our attention to one ("complex") dimensional domains. In this case, thanks to Theorem 1.1.5, we know that the domains then are complex manifolds.

Suppose (Σ, j) is a Riemann surface (i.e., Σ is a complex curve with complex structure j) and (M,J) is an almost complex manifold. A *smooth* map $u : \Sigma \to M$ is said to be J-holomorphic (or pseudo holomorphic) if its differential at every point is complex linear with respect to J, i.e.,

$$\mathrm{d}u \circ j = J \circ \mathrm{d}u. \tag{1.1}$$

Remark 1.2.1. To talk about derivatives, we need to at the very least assume that u is differentiable rather than smooth, but it will turn out later (see Section B.2.3), that u is as smooth as J allows it to be (atleast when u has some *minimum regularity*). We will therefore assume smoothness whenever convenient.

Eq. (1.1) is a *nonlinear* first order PDE, often called the *nonlinear Cauchy-Riemann equation*. Let us find an expression in local coordinates and see how it compares with the CR equation we know from elementary complex analysis. Consider a holomorphic chart $\varphi_\alpha : U_\alpha \to \mathbb{C}$ over $U_\alpha \subseteq \Sigma$. Since j is the complex structure for Σ, $\mathrm{d}\varphi_\alpha \circ j = i\,\mathrm{d}\varphi_\alpha$. Thus, $u : \Sigma \to M$ is J-holomorphic iff its local coordinate representation

$$u_\alpha := u \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \to M$$

is J-holomorphic with respect to the standard complex structure i on $\varphi_\alpha(U_\alpha)$.

In holomorphic coordinates $z = s + it$ on U_α, $J \circ \mathrm{d}u = \mathrm{d}u \circ j$ gives $J \circ \mathrm{d}u_\alpha = \mathrm{d}u_\alpha \circ i$, which translates to

$$\partial_s u_\alpha + J(u_\alpha)\partial_t u_\alpha = 0.$$

The word "non-linear" is used for the above CR equation, since $J(u_\alpha)$ is nonlinear in u_α. For a smooth map $u = f + ig : (\mathbb{C}, i) \to (\mathbb{C}^n, i = J_{\mathrm{std}})$, the above equation is equivalent to the CR equation in its familiar form $\partial_s f = \partial_t g$, $\partial_s g = -\partial_t f$. Thus i-holomorphic curves are holomorphic in the usual sense.

1.2.1 Ellipticity of the CR operator

Elliptic operators are a special sort of partial differential operators (PDOs) on a manifold, prevalent all across geometry and physics. Let us briefly introduce PDOs, to define ellipticity.

Let M be a manifold, and ∇ be a connection on TM. Let $f \in \mathscr{C}^\infty(M)$ be a real valued function, then the kth derivative of f using ∇ is denoted by $\nabla^k(f)$, and a partial derivative by $\nabla_{v_1 \ldots v_k} f := \nabla_{v_1} \ldots \nabla_{v_k} f$.

Definition 1.2.2. A PDO $\mathcal{P}$ on M of order k is an operator taking real functions f on M to real functions on M, depending on f and its first k derivatives. Explicitly, if f is a real function on M such that the first k derivatives $\nabla f, \ldots, \nabla^k f$ of f exist (possibly in some weak sense, see Section B.1), then $\mathcal{P}f$ is a real function on M given by

$$(\mathcal{P}u)(x) = \widetilde{\mathcal{P}}(x, f(x), \nabla f(x), \ldots, \nabla^k f(x))$$

for $x \in M$, where $\widetilde{\mathcal{P}}$ is some real function of its arguments. To make life easier, we assume $\widetilde{\mathcal{P}}$ is atleast continuous in all its arguments. $\mathcal{P}$ is said to be smooth, if $\widetilde{\mathcal{P}}$ is so (likewise for $\mathscr{C}^k$ and $\mathcal{W}^{k,p}$). If $\mathcal{P}f$ is linear in f, then $\mathcal{P}$ is called a linear PDO, otherwise non-linear.

It is often useful to regard a differential operator as a mapping between function spaces of various regularities ($\mathscr{C}^k(M)$, $\mathscr{C}^\infty(M)$, Holder spaces and Sobolev spaces). For instance, if $\mathcal{P}$ is a *smooth* PDO, then $\mathcal{P} : \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$. If $\mathcal{P}$ moreover has order k, then $\mathcal{P} : \mathscr{C}^{k+\ell}(M) \to \mathscr{C}^\ell(M)$. A standard problem in the theory of differential equations is to find solutions to equations of the form $\mathcal{P}f = 0$ or $\mathcal{P}f = v$ for a small perturbation v. A technique to simplify the nonlinear problem is to consider the linearized version. In this context, we introduce:

Definition 1.2.3. Let $\mathcal{P}$ be a PDO of order k, defined by a $\widetilde{\mathcal{P}}$ which is atleast $\mathscr{C}^1$ in the arguments $f, \nabla f, \dots, \nabla^k f$. Let $u \in \mathscr{C}^k(M)$. Define the *linearization* $L_u\mathcal{P}$ of $\mathcal{P}$ at u to be the derivative of $\mathcal{P}(v)$ with respect to v at u, that is

$$L_u\mathcal{P}v = \lim_{\alpha\to 0}\left(\frac{\mathcal{P}(u+\alpha v) - \mathcal{P}(u)}{\alpha}\right).$$

Then $L_u\mathcal{P}$ is a *linear* differential operator of order k. Note that even if $\mathcal{P}$ is a smooth operator, the linearization $L_u\mathcal{P}$ need not be smooth if u is not smooth. For instance, if $\mathcal{P}$ is of order k and $u \in \mathscr{C}^{k+\ell}(M)$, then $L_u\mathcal{P}$ will have $\mathscr{C}^\ell$ coefficients in general, as they depend on the kth derivatives of u.

Many properties of a linear PDO $\mathcal{P}$ depend only on the highest order derivatives occurring in $\mathcal{P}$ (and many properties of a nonlinear PDO arise from properties of its linearization). The *symbol* of $\mathcal{P}$ is a convenient way to isolate these higher order terms.

Definition 1.2.4. Let $\mathcal{P}$ be a linear PDO on functions of order k. Then, in index notation,

$$\mathcal{P}u = A^{i_1\dots i_k}\nabla_{i_1\dots i_k}u + B^{i_1\dots i_{k-1}}\nabla_{i_1\dots i_{k-1}}u + \cdots + K^{i_1}\nabla_{i_1}u + Lu \tag{1.2}$$

where $A, B, \dots, K$ are symmetric[1] tensors and L a real function on M [2]. For each point $x \in M$ and each $\xi \in T_x^*M$, with $\xi = \sum_i \xi_i \, dx^i$ locally, define $\sigma_\xi(\mathcal{P}; x) = A^{i_1\dots i_k}\xi_{i_1}\dots\xi_{i_k}$. Let $\sigma(\mathcal{P}) : T^*M \to \mathbb{R}$ be the function[3] with value $\sigma_\xi(\mathcal{P}; x)$ at each $\xi \in T_x^*M$. Then $\sigma(\mathcal{P})$ is called the *symbol* or *principal symbol* of $\mathcal{P}$. It is homogeneous polynomial of degree k on each cotangent space.

Now we can define linear elliptic operators on functions.

Definition 1.2.5. A PDO $\mathcal{P}$ of order k on M is said to be *elliptic* if for each $x \in M$ and each nonzero $\xi \in T_x^*M$, we have $\sigma_\xi(\mathcal{P}; x) \neq 0$.

These notions also hold for nonlinear PDOs, by considering the operator's linearization. Accordingly, for a nonlinear PDO $\mathcal{P}$, its symbol at u is the symbol of $L_u\mathcal{P}$, and $\mathcal{P}$ is elliptic at u if the $L_u\mathcal{P}$ is elliptic. A nonlinear operator $\mathcal{P}$ may be elliptic at some functions u and not at others.

The above notions can be extended to sections of vector bundles of appropriate regularities. The only significant difference is that in Eq. (1.2) the tensors $A^{i_1\cdots i_k}, B^{i_1\cdots i_k}, \dots$ won't be ordinary tensors. Instead, for a PDO $\mathcal{P}$ taking $\Gamma(V)$ to $\Gamma(W)$, they would be tensors taking values in $V^* \otimes W$, or equivalently, bundle homomorphisms from V to W. Likewise, the symbol would take values in bundle maps $\mathrm{Hom}(V, W)$. In such a case,

Definition 1.2.6. Let V, W be vector bundles over a manifold M, and let $\mathcal{P}$ be a linear PDO of degree k from $\Gamma(V)$ to $\Gamma(W)$. We say $\mathcal{P}$ is elliptic if for each $x \in M$ and each nonzero $\xi \in T_x^*M$, the linear map $\sigma_\xi(\mathcal{P}; x) : V_x \to W_x$ is *invertible*.

Now we arrive at the notion we want to consider for CR operators. Say that $\mathcal{P}$ is an *under-determined elliptic operator* if for each $x \in M$ and each $0 \neq \xi \in T_x^*M$, the map $\sigma_\xi(\mathcal{P}; x) : V_x \to W_x$ is *surjective*, and that P is an *over-determined elliptic operator* if for each $x \in M$ and each $0 \neq \xi \in T_x^*M$, the map $\sigma_\xi(\mathcal{P}; x) : V_x \to W_x$ is *injective*. Thus, in particular, if $\mathcal{P}$ is elliptic, then $\mathrm{rk}\, V = \mathrm{rk}\, W$, if $\mathcal{P}$ is *under-determined elliptic* then $\mathrm{rk}\, V \geq \mathrm{rk}\, W$, and $\mathcal{P}$ is *over-determined elliptic* if $\mathrm{rk}\, V \leq \mathrm{rk}\, W$.

Consider the equation $\mathcal{P}(v) = w$ in a small region of M. Locally we can think of v as a collection of ℓ real functions, and the equation $\mathcal{P}(v) = w$ as being m simultaneous equations on the ℓ functions of v. Now, guided by elementary linear algebra, we expect that a system of m equations in ℓ variables is likely to have many solutions if $\ell > m$ (under-determined), one solution if $\ell = m$, and no solutions at all if $\ell < m$ (over-determined).

[1]Why? Because for smooth functions, we have equality of mixed partial derivatives. So we better consider symmetric tensors.

[2]The notation given above is very concise. More elaborately, the principal symbol of a differential operator $\sum_{|\alpha|\leq k} a_\alpha(x)\partial_x^\alpha$ is the function $\sum_{|\alpha|=k} a_\alpha(x)\xi_i^\alpha$ where $\xi = (\xi_i)$ is a formal variable.

[3]We just want to keep track of a tuple of $A^{i_1\cdots i_k}$ and we are using function notation for that.

In the linear case, the symbol of the CR operator, being a first order homogeneous PDO, is just the matrix of coefficients. Let us elaborate on this. For a holomorphic function $f : \Omega \to \mathbb{C}$ on $\Omega \subseteq \mathbb{C}$, consider $f = (u, v)$, then the CR equation reduces to

$$\begin{pmatrix} \partial_x & -\partial_y \\ \partial_y & \partial_x \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0.$$

Thus the CR operator $\mathcal{P}$ is the matrix above and the symbol $\sigma(\mathcal{P}) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ at all points. More precisely, $\mathcal{P} : \mathscr{C}^\infty(\Omega, \mathbb{C}) \to \mathscr{C}^\infty(\Omega, \mathbb{C})$ is an operator between sections of the trivial vector bundle over Ω with fiber $\mathbb{C}$. It is given by $\mathcal{P}(u, v)^{\mathrm{t}} = \begin{pmatrix} \partial_x & -\partial_y \\ \partial_y & \partial_x \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$ where the matrix is the (vacuously, symmetric) tensor taking values in $\mathbb{C}^* \otimes \mathbb{C} \simeq \mathrm{Hom}(\mathbb{C}, \mathbb{C}) \simeq \mathbb{C}$. For every $x \in \Omega$ and every $\xi = \xi_1 \, \mathrm{d}x + \xi_2 \, \mathrm{d}y \in T_x^*\Omega$, the symbol is $\sigma_\xi(\mathcal{P}; x) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$. Ellipticity then follows from invertibility of the matrix.

The CR operator $\mathcal{P} : u \mapsto \partial_x u + J(u)\partial_y u$ is similarly elliptic. On functions into $\mathbb{C}$, say $u = (u_1, u_2)$, this would take the form of the operator

$$\mathcal{P} = \begin{pmatrix} \partial_x & 0 \\ 0 & \partial_x \end{pmatrix} + J(u) \begin{pmatrix} \partial_y & 0 \\ 0 & \partial_y \end{pmatrix}.$$

In the standard case $J(u) = J_{\mathrm{std}}$, this reduces to the equation above. We shall see that the CR operator is over-determined in (complex) dimensions greater than one. Hence, we don't expect existence of generic J-holomorphic submanifolds. Interestingly, there is a rich theory of existence of solutions to elliptic equations, and one of the regularity of the aforementioned solutions (see [5], section 1.4 for details).

1.2.2 Why only *curves*?

Let (X, j) be a complex manifold of complex dimension k and (M, J) be an almost complex manifold of real dimension $2n$. Further, suppose that $u : X \hookrightarrow M$ is a smooth J-holomorphic map. Thus, we have $\mathrm{d}u \circ j = J \circ \mathrm{d}u$. In local holomorphic coordinates $z_1, \dots, z_k$ with $z_j = x_j + i y_j$ on $U_\alpha \subseteq X$, $J \circ \mathrm{d}u = \mathrm{d}u \circ j$ gives $J \circ \mathrm{d}u_\alpha = \mathrm{d}u_\alpha \circ i$ (where u_α is the local representation for u), which translates to

$$\begin{aligned} J(u_\alpha) \cdot (\partial_{x_1} u_\alpha, \partial_{y_1} u_\alpha, \dots, \partial_{x_k} u_\alpha, \partial_{y_k} u_\alpha) &= (\partial_{x_1} u_\alpha, \partial_{y_1} u_\alpha, \dots, \partial_{x_k} u_\alpha, \partial_{y_k} u_\alpha) \circ i \\ &= (\partial_{y_1} u_\alpha, \partial_{x_1} u_\alpha, \dots, \partial_{y_k} u_\alpha, \partial_{x_k} u_\alpha) \end{aligned}$$

which gives us the system

$$\partial_{x_j} u_\alpha + J(u_\alpha) \partial_{y_j} u_\alpha = 0 \ \forall \ j.$$

Since we can consider each coordinate separately, assume WLOG $n = 1$. Thus we have the system of PDOs

$$\mathcal{P}_j = \begin{pmatrix} \partial_{x_j} & 0 \\ 0 & \partial_{x_j} \end{pmatrix} + J(u) \begin{pmatrix} \partial_{y_j} & 0 \\ 0 & \partial_{y_j} \end{pmatrix} \quad \forall \ j.$$

This system can be combined into a grand PDO $\mathcal{P} : \mathscr{C}^\infty(U_\alpha, \mathbb{C}) \to \mathscr{C}^\infty(U_\alpha, \mathbb{C}^n)$ (assuming J is standard), given as

$$\begin{pmatrix} \partial_{x_1} & \partial_{y_1} & \dots & \partial_{x_k} & \partial_{y_k} \\ -\partial_{y_1} & \partial_{x_1} & \dots & -\partial_{y_k} & \partial_{x_k} \end{pmatrix}^t.$$

The symbol of the above operator is the matrix

$$\begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ -1 & 1 & \dots & -1 & 1 \end{pmatrix}^t$$

which is injective since the two rows above are linearly independent. Thus, the above system is an over-determined elliptic system of linear PDEs for $k \geq 2$, and so generically there are no solutions. As a result, we study J-holomorphic submanifolds only of complex dimension 1.

1.3 Contrast with honest holomorphic curves

The nonlinear CR equation being elliptic, J-holomorphic curves share many of the nice properties of holomorpihc curves in a complex manifold.

1.3.1 Unique continuation

In the case where J is integrable, since every holomorphic function (of one complex variable) admits a power series expansion, we have an identity principle. Here we give an analogue of this principle in the almost complex case.

Since the question is local, we take the domain of u to be a ball $B_\varepsilon := \{z \in \mathbb{C} \mid |z| < \varepsilon\}$ in $\mathbb{C}$ and M to be $\mathbb{C}^n$. Hence, J is a function on $\mathbb{C}^n$ with values in $GL(2n, \mathbb{R})$ such that $J^2 = -\mathbb{1}$. For later reference, it is convenient to prove the unique continuation theorem under the weakest possible smoothness hypotheses on J. Thus we shall consider functions $u : B_\varepsilon \to \mathbb{C}^n$ that satisfy the equation $\partial_s u + J(u)\partial_t u = 0$, where $J : \mathbb{C}^n \to GL(2n, \mathbb{R})$ is a $\mathscr{C}^1$-function.

Remark 1.3.1. Consider the standard Laplacian $\Delta = \partial_s^2 + \partial_t^2$. Since $\partial_t(J^2) = (\partial_t J)J + J(\partial_t J) = 0$, we have that every $\mathcal{W}^{1,p}$-solution u of the local CR equation with $p > 2$ is also a weak solution of the second order quasi-linear equation

$$\Delta u = (\partial_t J(u))\partial_s u - (\partial_s J(u))\partial_t u. \tag{1.3}$$

An elliptic bootstrapping argument (Section B.2.3) based on the above equation then shows that if J is of class $\mathscr{C}^\ell$ for some $\ell \geq 1$ then every J-holomorphic curve u of class $\mathcal{W}^{1,p}$ with $p > 2$ is necessarily of class $\mathcal{W}^{\ell+1,q}$ for every $q < \infty$.

A smooth function $w : B_\varepsilon \to \mathbb{C}^n$ is said to **vanish to infinite order** at 0 if its ∞-jet[4] vanishes at order 0. To capture this property for arbitrary functions which are not necessarily (even weakly) differentiable, we use the following reasonable approximation. An integrable function $w : B_\varepsilon \to \mathbb{C}^n$ is said to vanish to infinite order at 0 if

$$\int_{|z|\leq r} |w(z)| = O(r^k) \qquad \forall\, k > 0.$$

The integral form of this condition is meaningful for every integrable function, and implies the earlier condition for smooth maps.

Theorem 1.3.2 (Unique continuation). Suppose that $u, v \in \mathscr{C}^1(B_\varepsilon, \mathbb{C}^n)$ satisfy the CR equation for some $\mathscr{C}^1$ function $J : \mathbb{C}^n \to GL(2n, \mathbb{R})$, and that $u - v$ vanishes to infinite order at zero. Then $u \equiv v$.

Corollary 1.3.3. Assume Σ is a connected Riemann surface and J is a $\mathscr{C}^1$ almost complex structure on M. If two J-holomorphic curves $u, v : \Sigma \to M$ agree to infinite order at a point $z \in \Sigma$, then $u \equiv v$.

Proof. By Theorem 1.3.2, the set of all points $z \in \Sigma$ such that u and v agree to infinite order at z is open and closed. □

Aronszajn's theorem. The function $w = v - u$ is of class $\mathcal{W}^{2,p}$ (by the elliptic bootstrapping mentioned in Remark 1.3.1) for every $p < \infty$ and vanishes to infinite order at zero. In particular, w is continuously differentiable. Because J and its derivatives are bounded (on a precompact smaller ball say), it follows from Eq. (1.3) that w satisfies a differential inequality of the form

$$|\Delta w(z)| \leq c(|w| + \partial_s w| + |\partial_t w|) \quad \forall\, z = s + it \in B_\varepsilon. \tag{1.4}$$

Hence Theorem 1.3.2 follows from Aronszajn's unique continuation theorem which we now quote:

Theorem 1.3.4 (Aronszajn). Let $\Omega \subseteq \mathbb{C}$ be a connected open set. Suppose that the function $w \in \mathcal{W}^{2,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ satisfies the pointwise estimate Eq. (1.4) (almost everywhere) in Ω and that w vanishes to infinite order at some point $z_0 \in \Omega$. Then $\omega \equiv 0$.

An alternative proof is given by the Carleman similarity principle covered in the next section.

[4]The k-jet of a (sufficiently differentiable) function f is defined as the Taylor series of f truncated till order k.

1.3.2 Carleman similarity principle

The most useful tool in understanding the regularity properties of J-holomorphic curves for general non-smooth J is the Carleman similarity principle. It says that one can transform any solution u of the perturbed nonlinear Cauchy Riemann equation

$$\partial_s u(z) + J(z)\partial_t u(z) + C(z)u(z) = 0 \tag{1.5}$$

into a holomorphic function by multiplication by a suitable matrix valued function $z \mapsto \Phi(z)$.

Theorem 1.3.5. Let $p > 2, C \in \mathcal{L}^p(B_\varepsilon, \mathbb{R}^{2n\times 2n})$ and $J \in \mathcal{W}^{1,p}(B_\varepsilon, \mathbb{R}^{2n\times 2n})$ such that $J^2 = -\mathbb{1}$. Suppose that $u \in \mathcal{W}^{1,p}(B_\varepsilon, \mathbb{R}^{2n})$ is a solution of Eq. (1.5) such that $u(0) = 0$. Then there is a $\delta \in (0, \varepsilon)$, a map $\Phi \in \mathcal{W}^{1,p}(B_\delta, \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}^n, \mathbb{R}^{2n})$, and a holomorphic map $\sigma : B_\delta \to \mathbb{C}^n$ such that $\Phi(z)$ is invertible and

$$u(z) = \Phi(z)\sigma(z), \quad \sigma(0) = 0, \quad \Phi(z)^{-1}J(z)\Phi(z) = i \quad \text{for every } z \in B_\delta.$$

Geometrically, one can think of the map $z \mapsto \Phi^{-1}(z)$ as a further trivialization (change of coordinates) for the bundle $(B_\delta \times \mathbb{R}^{2n}, J)$ that converts the varying family of almost complex structures $J(z)$ on the fibers $z \times \mathbb{R}^{2n}$ into the constant family i, and that transforms the section $\widetilde{u} : z \mapsto (z, u(z))$ into a holomorphic section $\widetilde{\sigma} : z \mapsto (z, \sigma(z))$. A proof involves reducing the theorem to the case when $J(z) = i$ and C is i-linear (see [2], pp. 24).

1.3.3 Critical points

A **critical point** (or singular point) of a curve $u : \Sigma \to M$ is a point $z \in \Sigma$ with $\mathrm{d}u(z) = 0$. The image $u(z)$ is then called a **critical value** of u. If we think of the image $C = u(\Sigma) \subseteq M$ as an unparametrized curve, then a critical point on C is a critical value of u. Points on C which are not critical are called regular or nonsingular. In the integrable case, critical points of nonconstant curves are well known to be isolated. The following lemma asserts this for arbitrary almost complex structures.

Lemma 1.3.6. Let Σ be a closed Riemann surface, J be a $\mathscr{C}^1$ almost complex structure on M, and $u : \Sigma \to M$ be a non-constant J-holomorphic curve. Then the set

$$X := u^{-1}\{u(z) \mid z \in \Sigma,\, \mathrm{d}u(z) = 0\}$$

of preimages of critical values is finite. Moreover, $u^{-1}(x)$ is a finite set for every $x \in M$.

Proof (assuming J is smooth and u is weakly differentiable). It suffices to prove that critical points are isolated, and so we may work locally. Thus we assume that $\Omega \subseteq \mathbb{C}$ is an open neighborhood of zero, that $u : \Omega \to \mathbb{C}^n$ is J-holomorphic for some $\mathscr{C}^1$ almost complex structure $J : \mathbb{C}^n \to GL(2n, \mathbb{R})$, and that

$$u(0) = 0, \quad \mathrm{d}u(0) = 0, \quad u \not\equiv 0, \quad J(0) =: J_0.$$

Write $z = s + it$. Since u is weakly differentiable, it is $\mathcal{W}^{1,\infty}$ on compact Σ, so by elliptic bootstrapping, u is smooth. Since u is non-constant, it follows from Corollary 1.3.3 that the ∞-jet of u at $z = 0$ must be nonzero. Hence there exists an integer $\ell \geq 2$ such that $u(z) = O(|z|^\ell)$ and $u(z) \neq O(|z|^{\ell+1})$. This implies $J(u(z)) = J_0 + O(|z|^\ell)$ (as $J(u(z)) - J_0 = (J'(u(z(t')))u'(z(t'))z'(t'))\,(u(z) - u(0))$ for some t' and the scaling function can be bounded; now we just use equivalence of norms on $\mathbb{C}$). Now examine the Taylor expansion of $\partial_s u + J(u)\partial_t u = 0$ *up to order* $\ell - 1$ to obtain

$$\partial_s T_\ell(u) + J_0 \partial_t T_\ell(u) = 0.$$

Here $T_\ell(u)$ denotes the Taylor expansion of u up to order ℓ (note that over order ℓ, the result is not the CR equation of $T_\ell(u)$, rather some combination of $\ell z^{\ell-1}$'s; the above is cleverly set up since the Taylor series of u begins at z^ℓ). It follows that $T_\ell(u) : \mathbb{C} \to \mathbb{C}^n$ is a holomorphic function (it satisfies the CR equation above) and there exists a nonzero vector $a \in \mathbb{C}^n$ such that

$$u(z) = az^\ell + O(|z|^{\ell+1}), \qquad \partial_s u(z) = \ell a z^{\ell-1} + O(|z|^\ell).$$

Hence $0 < |z| \leq \varepsilon \implies u(z) \neq 0$, $\mathrm{d}u(z) \neq 0$ with $\varepsilon > 0$ sufficiently small. Hence critical points are isolated and, since Σ is compact, the set of critical points of u is finite. It also follows that $u^{-1}(p)$ is finite set for every $p \in M$. Hence X is a finite set. $\square$

Proof (using Carleman similarity). Assume $J : \mathbb{C}^n \to GL(2n,\mathbb{R})$ is an almost complex structure of class $\mathscr{C}^1$. Let $\Omega \subseteq \mathbb{C}$ be an open set. Then every J-holomorphic curve $u : \Omega \to \mathbb{C}^n$ is of class $\mathcal{W}^{2,p}$ for every $p < \infty$. Suppose u is such a J-holomorphic curve. Then u satisfies Eq. (1.5) where $J(z) = J(u(z))$ is of class $\mathscr{C}^1$ and $C(z) = 0$. Hence it follows from Theorem 1.3.5, that $u^{-1}(p)$ is a finite set for every $p \in M$. Moreover, differentiating Eq. (1.5) with $C = 0$, we find that the function $v := \partial_s u \in \mathcal{W}^{1,p}(\Omega, \mathbb{C}^n)$ satisfies the equation

$$\partial_s v(z) + J(z)\partial_t v(z) + (\partial_s J(z))J(z)v(z) = 0.$$

Applying Theorem 1.3.5 to v we see that its zeros are isolated. Hence the set of critical points of u is finite and so is the set X. □

We now show how to choose nice coordinates near a regular point of a J-holomorphic curve.

Lemma 1.3.7. Let $\ell \geq 2$ and J be a $\mathscr{C}^\ell$ almost complex structure on a smooth manifold M. Let $\Omega \subseteq \mathbb{C}$ be an open neighborhood of zero and $u : \Omega \to M$ be a local J-holomorphic curve such that $\mathrm{d}u(0) \neq 0$. Then there exists a $\mathscr{C}^{\ell-1}$ coordinate chart $\psi : U \to \mathbb{C}^n$, defined on an open neighborhood of $u(0)$, such that

$$\psi \circ u(z) = (z, 0, \ldots, 0), \qquad \mathrm{d}\psi(u(z))J(u(z)) = J_{\mathrm{std}}\,\mathrm{d}\psi(u(z)) \quad \text{for } z \in \Omega \cap u^{-1}(U).$$

Proof. By elliptic regularity (see Remark 1.3.1), u is of class $\mathcal{W}^{\ell+1,p}$ for every $p < \infty$ and hence, by the Sobolev embedding theorem (Theorem B.1.2), is of class $\mathscr{C}^\ell$. Write $z = s + it \in \Omega$ and $w = (w_1, w_2, \ldots, w_n) \in \mathbb{C}^n$ where $w_j = x_j + iy_j$. Shrink Ω if necessary and choose a complex $\mathscr{C}^{\ell-1}$ frame of the bundle u^*TM[5] such that

$$Z_1(z), \ldots, Z_n(z) \in T_{u(z)}M, \qquad Z_1 = \partial u/\partial z.$$

Define $\phi : \Omega \times \mathbb{C}^{n-1} \to M$ by

$$\phi(w_1, \ldots, w_n) = \exp_{u(w_1)}\left(\sum_{j=2}^{n} x_j Z_j(w_1) + \sum_{j=2}^{n} y_j J(u(w_1))Z_j(w_1)\right).$$

Then ϕ is a $\mathscr{C}^{\ell-1}$ diffeomorphism of a (normal) neighborhood V of zero in $\mathbb{C}^n$ onto a neighborhood U of $u(0)$ in M. It satisfies $\phi(z_1, 0, \ldots, 0) = u(z_1)$ and

$$\frac{\partial \phi}{\partial x_j} + J(\phi)\frac{\partial \phi}{\partial y_j} = 0, \qquad j = 1, \ldots, n,$$

at all points $z = (z_1, 0, \ldots, 0)$. Hence the inverse $\psi = \phi^{-1} : U \to V$ has the required properties. □

We next start investigating the intersections of two distinct J-holomorphic curves. We prove a useful result asserting that intersection points of distinct J-holomorphic curves $u : \Sigma \to M$ and $u' : \Sigma' \to M$ can only accumulate at points which are critical on both curves $C = u(\Sigma)$ and $C' = u'(\Sigma')$. For local J-holomorphic curves this statement can be reformulated as:

Lemma 1.3.8. Let J be a $\mathscr{C}^2$ almost complex structure on M. Let $\Omega \subseteq \mathbb{C}$ be an open neighborhood of zero and $u, v : \Omega \to M$ be J-holomorphic curves such that

$$u(0) = v(0), \qquad \mathrm{d}u(0) \neq 0.$$

Moreover, assume that there exist sequences $z_\nu, \zeta_\nu \in \Omega$ such that

$$u(z_\nu) = v(\zeta_\nu), \qquad \lim_{\nu\to\infty} z_\nu = \lim_{\nu\to\infty} \zeta_\nu = 0, \qquad \zeta_\nu \neq 0.$$

Then there exists a holomorphic function $\phi : B_\varepsilon(0) \to \Omega$ defined in some neighborhood of zero such that $\phi(0) = 0$ and $v = u \circ \phi$. Thus, u and v are the same curve up to reparametrization.

[5] Since u is only $\mathscr{C}^\ell$, u^*TM is a $\mathscr{C}^{\ell-1}$-vector bundle.

Proof (assuming that J is smooth and u is weakly differentiable). By Lemma 1.3.7, we may assume that $M=\mathbb{C}^n$, $J:\mathbb{C}^n\to GL(2n,\mathbb{R})$ is smooth, and

$$u(z)=(z,0),\qquad J(w_1,0)=i$$

where $w=(w_1,\widetilde{w})$ with $\widetilde{w}\in\mathbb{C}^{n-1}$. Write $v(z)=:(v_1(z),\widetilde{v}(z))$.

We show first that the ∞-jet of $\widetilde{v}$ at $z=0$ must vanish. Otherwise there would exist an integer $\ell\geq 0$ such that $\widetilde{v}(z)=O(|z|^\ell)$ and $\widetilde{v}(z)\neq O(|z|^{\ell+1})$. As in the proof of Lemma 1.3.6, consider the Taylor expansion up to order $\ell-1$ on the left hand side of the equation $\partial_s v+J\partial_t v=0$ to obtain that $T_\ell(v)$ is holomorphic. Hence

$$v_1(z)=p(z)+O(|z|^{\ell+1}),\qquad \widetilde{v}(z)=\widetilde{a}z^\ell+O(|z|^{\ell+1})$$

where $p(z)$ is a polynomial of order ℓ and $a\in\mathbb{C}^{n-1}$ is nonzero. This implies that $\widetilde{v}(\zeta)\neq 0$ in some neighborhood of zero and hence $v(\zeta)\notin\operatorname{im}u$ for every nonzero element ζ in this neighborhood, in contradiction to the assumption of the lemma. Thus we have proved that the ∞-jet of $\widetilde{v}$ at $z=0$ vanishes.

We prove that $\widetilde{v}(z)\equiv 0$. To see this note that, because $J=J_{\mathrm{std}}$ along the axis $\widetilde{w}=0$, we have

$$\frac{\partial J(w_1,0)}{\partial x_1}=\frac{\partial J(w_1,0)}{\partial y_1}=0,$$

for all w_1. Hence, from $|\partial J(w)/\partial x_1-\partial J(w_1,0)/\partial x_1|\leq|\sup_w J'(w)||\widetilde{w}|$, etc.,

$$\left|\frac{\partial J(w)}{\partial x_1}\right|+\left|\frac{\partial J(w)}{\partial y_1}\right|\leq c|\widetilde{w}|.$$

Using Eq. (1.3) we obtain an inequality

$$|\Delta\widetilde{v}|\leq c(|\widetilde{v}|+|\partial_s\widetilde{v}|+|\partial_t\widetilde{v}|).$$

Hence it follows from Aronszajn's theorem that $\widetilde{v}\equiv 0$. Thus $v(z)=(v_1(z),0)$ which implies that $\phi:=v_1$ is holomorphic and $v(z)=u(\phi(z))$ for $z\in B_\delta$ as required. □

Proof (using Carleman similarity). In this proof we consider the general case where J is of class $\mathscr{C}^\ell$, for some constant $\ell\geq 2$, and use the Carleman similarity principle. By Lemma 1.3.7, we may assume that $J:\mathbb{C}^n\to GL(2n,\mathbb{R})$, $u(z)=(z,0)$ and $v=(v_1,\widetilde{v})$ are as in the first proof, except that now J and v are only continuously differentiable. We claim that there exists a continuous function $\widetilde{C}:\Omega\to\mathrm{End}_{\mathbb{R}}(\mathbb{C}^{n-1})$ such that

$$\partial_s\widetilde{v}+i\partial_t\widetilde{v}+\widetilde{C}\widetilde{v}=0.$$

This equation can be written in the form $\widetilde{C}\widetilde{v}=\widetilde{\pi}((J(v)-i)\partial_t v)$, where $\widetilde{\pi}:\mathbb{C}^n\to\mathbb{C}^{n-1}$ denotes the projection $\widetilde{\pi}(w_1,\widetilde{w}):=\widetilde{w}$. Since $J(v_1,0)=i$, the function $\widetilde{C}$ is given by

$$\widetilde{C}(z)\widetilde{\zeta}=\widetilde{\pi}\left(\left(\int_0^1\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0}J(v_1(z),\tau\widetilde{v}(z)+s\widetilde{\zeta})\,\mathrm{d}\tau\right)\partial_t v(z)\right)$$

for $z\in\Omega$ and $\widetilde{\zeta}\in\mathbb{C}^{n-1}$. Note that $\widetilde{C}$ is continuous.

It now follows from Theorem 1.3.5, that there is a constant $\delta>0$, a function $\widetilde{\Phi}\in\mathcal{W}^{1,p}(B_\delta,GL(n-1,\mathbb{C}))$, and a holomorphic function $\widetilde{\sigma}:B_\delta\to\mathbb{C}^{n-1}$ such that

$$\widetilde{v}(z)=\widetilde{\Phi}(z)\widetilde{\sigma}(z),\qquad\widetilde{\sigma}(0)=0.$$

By assumption, there is a sequence $0\neq\zeta_\nu\to 0$ such that $\widetilde{v}(\zeta_\nu)=0$. Hence $\widetilde{\sigma}(\zeta_\nu)=0$ for ν sufficiently large and this implies $\widetilde{v}\equiv 0$. □

1.3.4 Somewhere injective curves

Let (Σ, j) be a closed Riemann surface and (M, J) be an almost complex manifold. A J-holomorphic curve $u : \Sigma \to M$ is said to be **mulitply covered** if there exists a closed Riemann surface (Σ', j'), a J-holomorphic curve $u' : \Sigma' \to M$, and a holomorphic branched covering $\phi : \Sigma \to \Sigma'$ such that

$$u = u' \circ \phi, \qquad \deg(\phi) > 1.$$

The curve u is called **simple** if it is not multiply covered. We shall see in the next chapter that the simple J-holomorphic curves in a given homology class form a a smooth finite dimensional manifold for generic J. In other words, the multiply covered curves are the exceptional case and may be singular points in the moduli space of all J-holomorphic curves. The proof of this result is based on the observation that every J-holomorphic curve is **somewhere injective** in the sense that

$$\mathrm{d}u(z) \neq 0, \qquad u^{-1}(u(z)) = \{z\}$$

for some $z \in \Sigma$. A point $z \in \Sigma$ with this property is called an **injective point** of u. Let us denote by

$$Z(u) := \{z \in \Sigma \mid \mathrm{d}u(z) = 0 \text{ or } \#u^{-1}(u(z)) > 1\}$$

the complement of the set of injective points. The next result shows that the set of injective points is open and dense for every simple J-holomorphic curve.

Proposition 1.3.9. Let J be a $\mathscr{C}^2$ almost complex structure on M, Σ be a closed Riemann surface, and $u : \Sigma \to M$ be a simple J-holomorphic curve. Then u is somewhere injective. Moreover, the set $Z(u)$ of non-injective points is at most countable and can only accumulate at the critical points of u.

Proof. We show that any curve $u : \Sigma \to M$ may be expressed as a composition $u' \circ \phi : \Sigma \to \Sigma' \to M$ where $u' : \Sigma' \to M$ is somewhere injective. If u is simple, ϕ must have degree 1 and it follows that u is also somewhere injective. This proof constructs the domain Σ' from the image curve $u(\Sigma)$ in M. For another proof, see [2], pp. 31.

Let $X \subseteq \Sigma$ denote the set of preimages of critical values of u and $X' = u(X)$ the set of critical values. By Lemma 1.3.6, these sets are finite. Let Q be the set of points in $Y = u(\Sigma) \setminus X'$ where distinct branches of $u(\Sigma)$ meet. This means $x \in Q$ iff there exist points $z_1 \neq z_2 \in \Sigma \setminus X$ such that

$$u(z_1) = u(z_2) =: x, \qquad u(U_1) \neq u(U_2) \quad \text{and thus} \quad u(U_1 \setminus \{z_1\}) \cap u(U_2 \setminus \{z_2\}) = \emptyset \ \text{(by Lemma 1.3.8)}$$

for two sufficiently small neighborhoods U_j of z_j. By Lemma 1.3.8, Q is a discrete subset of Y (i.e., it has no accumulation points in Y). Thus the set

$$S = Y \setminus Q$$

is an embedded submanifold in M. Let $\iota : S \to M$ denote this embedding. Since only finitely many branches of Y can meet at each point of Q, each such point gives rise to a finite number of ends of S each diffeomorphic to a punctured disc $D \setminus \{\mathrm{pt}\}$. Therefore, we may add a point to each of these ends and extend ι smoothly over the resulting surface S' to ι'. Because ι' is an immersion, there is a unique complex structure on S' with respect to which ι' is J-holomorphic.

The manifold S' still has ends corresponding to the points in X'. Each such end corresponds to a distinct branch of $\operatorname{im} u$ through a point in X'. Thus it is the conformal image of a punctured disc, and so must have the conformal structure of the punctured disc. Therefore we may form a closed Riemann surface Σ' by adding a point to each end of S'. Further, because u extends over the whole of Σ, the map ι' must extend to a J-holomorphic map $u' : \Sigma' \to M$. The map u' is somewhere injective and u factors as $u' \circ \phi$ where ϕ is a holomorphic map $\Sigma \to \Sigma'$. Thus ϕ is a branched cover and has degree one iff u is somewhere injective. □

1.4 Energy identity

In this section, we introduce a crucial notion in the development of Gromov's compactness theory for J-holomorphic curves. Let (M, ω) be an almost symplectic manifold (i.e., ω is non-degenerate but not necessarily closed), $J \in \mathcal{J}_\tau(M)$ be an ω-tame almost complex structure, and $(\Sigma, j_\Sigma, \mathrm{dvol}_\Sigma)$ be a compact Riemann surface. Then M

carries a metric g_J determined by ω and J, and the Riemann surface carries a metric determined by j_Σ and dvol_Σ. A smooth map $u : \Sigma \to M$ is a J-holomorphic curve iff it is conformal with respect to g_J, i.e., its differential preserves angles, or equivalently, it preserves inner products up to a common positive factor.

The **energy** of a smooth map $u : \Sigma \to M$ is defined as the square of the L^2-norm of the 1-form $\mathrm{d}u \in \Omega^1(\Sigma, u^*TM)$:

$$E(u) = \frac{1}{2}\int_\Sigma |\,\mathrm{d}u|^2 \mathrm{dvol}.$$

Here the norm of the $\mathbb{R}$-linear map $L = \mathrm{d}u(z) : T_z\Sigma \to T_{u(z)}M$ is defined by

$$|L|_J := |\zeta|^{-1}\sqrt{|L(\zeta)|_J^2 + |L(j_\Sigma\zeta)|_J^2}$$

for $0 \neq \zeta \in T_z\Sigma$. Here, the norms are taken with respect to the metrics acquired above. The right side can be easily shown to be independent of ζ.

It is important to note that, while the energy density $|\,\mathrm{d}u|_J^2$ depends on the metric on Σ, the energy $E(u)$ depends only on the complex structure j_Σ and not on the volume form. To see this, note that

$$|\alpha|^2 \mathrm{dvol}_\Sigma = \alpha \wedge *\alpha = -\alpha \wedge (\alpha \circ j_\Sigma)$$

for every Hermitian vector bundle $E \to \Sigma$ and every 1-form $\alpha \in \Omega^1(\Sigma, E)$. For general smooth maps $u : \Sigma \to M$ the energy $E(u)$ also depends on the metric g_J on M. However, we now show that in the case of J-holomorphic curves in symplectic manifolds the energy is a *topological* invariant that depends only on the homology class of the curve.

Lemma 1.4.1. Let ω be a non degenerate 2-form on a smooth manifold M. If J is ω-tame then every J-holomorphic curve $u : \Sigma \to M$ satisfies the energy identity

$$E(u) = \int_\Sigma u^*\omega.$$

If J is ω-compatible then every smooth map $u : \Sigma \to M$ satisfies

$$E(u) = \int_\Sigma |\overline{\partial}_J(u)|^2 \mathrm{dvol}_\Sigma + \int_\Sigma u^*\omega.$$

Proof. By choosing conformal coordinates $z = s + it$ we may assume that Σ is an open subset of $\mathbb{C}$. In this case,

$$\begin{aligned}
\frac{1}{2}|\,\mathrm{d}u|_J^2 \mathrm{dvol}_\Sigma &= \frac{1}{2}\left(|\partial_s u|_J^2 + |\partial_t u|_J^2\right) \mathrm{d}s \wedge \mathrm{d}t \\
&= \frac{1}{2}|\partial_s u + J\partial_t u|_J^2 \,\mathrm{d}s \wedge \mathrm{d}t - \langle \partial_s u, J\partial_t u\rangle_J \,\mathrm{d}s \wedge \mathrm{d}t \\
&= |\overline{\partial}_J(u)|_J^2 \mathrm{dvol}_\Sigma + \frac{1}{2}\left(\omega(\partial_s u, \partial_t u) + \omega(J\partial_s u, J\partial_t u)\right) \mathrm{d}s \wedge \mathrm{d}t.
\end{aligned}$$

In the ω-compatible case the last term on the right is equal to $u^*\omega$. If J is ω-tame and $\partial_s u + J\partial_t u = 0$ then the first term on the right vanishes and the last term on the right is also equal to $u^*\omega$. □

Lemma 1.4.1 shows that if ω is closed, J is ω-compatile and Σ is closed, then J-holomorphic curves minimize energy in their homology class, and hence are harmonic maps. This need not be the case if J is only ω-tame. However in both cases the energy of a J-holomorphic curve is a topological invariant provided that ω is closed. Identifying area with energy also shows that J-holomorphic curves are *minimal surfaces* (that is, area minimizing surfaces), which will have important consequences in the compactness theory developed later (Theorem 4.1.2).

Chapter 2

Moduli spaces of simple curves

Due to the properties of the CR operator, holomorphic curves in a closed symplectic manifold typically occur in finite dimensional smooth moduli spaces. In the first section, we outline the construction of this moduli space and present the proof in the second section. The last section deals with a lemma which would be useful to argue regularity for a CR operator later on. Here, we shall restrict ourselves to the study of parametrized simple curves i.e., those defined on a Riemann surface *with a fixed complex structure* j_Σ.

2.1 The moduli space

Let (M^{2n},ω) be a closed symplectic manifold, (Σ, j_Σ) be a closed Riemann surface and J be an ω-tame almost complex structure. Consider the equation $\overline{\partial}_J(u)=0$, where

$$\overline{\partial}_J(u)=\frac{1}{2}(\,\mathrm{d}u+J\circ\,\mathrm{d}u\circ j_\Sigma).$$

Since we shall later on consider curves with certain energy constraints (see Section 1.4) and to simplify the problem, we shall restrict ourselves to the moduli space of solutions of the above CR equation that represents the homology class $A\in H_2(M;\mathbb{Z})$. That is, the object of our focus would be

$$\mathcal{M}(\Sigma,A;J)=\{u\in\mathscr{C}^\infty(\Sigma,M)\mid J\circ\,\mathrm{d}u=\,\mathrm{d}u\circ j_\Sigma,[u]=A\}.$$

The subspace of simple solutions to the above delbar equation is particularly nice, and we denote it by $\mathcal{M}^*(\Sigma,A;J)$.

The moduli space $\mathcal{M}(A,\Sigma;J)$ can be interpreted as the zero set of the $\overline{\partial}_J$ section introduced above. Let $\mathcal{B}\subseteq\mathscr{C}^\infty(\Sigma,M)$ (respectively, $\mathscr{C}^k(\Sigma,M)$) denote the space of all smooth (resp., $\mathscr{C}^k$) maps $u:\Sigma\to M$ that represent the class $A\in H_2(M;\mathbb{Z})$. This space can be thought of as an infinite dimensional Fréchet (resp., Banach) manifold whose tangent space at $u\in\mathcal{B}$ is the space

$$T_u\mathcal{B}=\Omega^0(\Sigma,u^*TM)$$

of all smooth vector fields $\xi(z)\in T_{u(z)}M$ along u. Consider the infinite dimensional vector bundle $\mathcal{E}\to\mathcal{B}$, whose fiber at u is the space

$$\mathcal{E}_u=\Omega^{0,1}(\Sigma,u^*TM)$$

of smooth J-antilinear 1-forms on Σ with values in u^*TM. Then the complex antilinear part of $\mathrm{d}u$ defines a section $\mathcal{S}:\mathcal{B}\to\mathcal{E}, u\mapsto(u,\overline{\partial}(u))$. The moduli space $\mathcal{M}(A,\Sigma;J)$ is the zero set of this section. The subset $\mathcal{M}^*(A,\Sigma;J)$ is the intersection of $\mathcal{M}(A,\Sigma;J)$ with the open set $\mathcal{B}^*\subseteq\mathcal{B}$ of all smooth maps $u:\Sigma\to M$ that represent the class A and are somewhere injective.

The operator D_u. To say that the above zero set is a manifold, we take hints from the finite dimensional regular level set theorem, and show that $\overline{\partial}$ is transverse to the zero section. This means that the image of the differential $\mathrm{d}\mathcal{S}(u):T_u\mathcal{B}\to T_{(u,0)}\mathcal{E}$ is complementary to the tangent space $T_u\mathcal{B}$ of the zero section for every $u\in\mathcal{M}^*(A,\Sigma;J)$. Given $u\in M^*(A,\Sigma;J)$ we define D_u

$$D_u:=D_{J,u}:=D\mathcal{S}(u):\Omega^0(\Sigma,u^*TM)\to\Omega^{0,1}(\Sigma,u^*TM)$$

to be the composite of linearization of the $\overline{\partial}$ operator $\mathrm{d}\overline{\partial}(u):T_u\mathcal{B}\to T_{(u,0)}\mathcal{E}$ with the projection $\pi_u:T_{(u,0)}\mathcal{E}=T_u\mathcal{B}\oplus\mathcal{E}_u\to\mathcal{E}_u$ onto the vertical component (a splitting of the tangent space exists along the zero section without the need of a connection on $\mathcal{E}$). The operator D_u is called the **vertical differential** of the section $\mathcal{S}$ at u. Transversality can now be expressed in the form that the linearized operator D_u is surjective for every $u\in\mathcal{M}^*(A,\Sigma;J)$.

In local coordinates on Σ and M, a J-holomorphic curve is a solution $u : \mathbb{C} \to \mathbb{R}^{2n}$ of the equation $\partial_s u + J(u)\partial_t u = 0$ and a vector field along u is a map $\xi : \mathbb{C} \to \mathbb{R}^{2n}$. A local formula for $D_u\xi$ is obtained by differentiating the LHS of this equation in the direction ξ. Thus,

$$D_u\xi = \eta\, \mathrm{d}s - J(u)\eta\, \mathrm{d}t, \qquad \eta := \frac{1}{2}(\partial_s\xi + J(u)\partial_t\xi + \partial_\xi J(u)\partial_t u).$$

Using the fact that u is J-holomorphic, we find that

$$D_u\xi = \overline{\partial}_J\xi - \frac{1}{2}(J\partial_\xi J)(u)\partial_J(u).$$

Note that the second term on the right hand side is an anti holomorphic 1-form because

$$\partial_J(u) := \frac{1}{2}(\,\mathrm{d}u - J \circ \mathrm{d}u \circ j)$$

is a holomorphic 1-form and J anti commutes with $J(\partial_\xi J)$. The above discussion shows that the operator defined by this formula is independent of the choice of local coordinates.

In Theorem C.1.3, we shall note that D_u is a real linear Cauchy-Riemann operator and hence is Fredholm, with index given by the Riemann Roch theorem,

$$\mathrm{index}(D_u) = n(2-2g) + 2c_1(u^*TM),$$

where g is the genus of Σ. The complex antilinear part of D_u is a lower order (compact) perturbation and so can be removed without changing the Fredholm index. The main part of D_u is the complex linear part is the complex linear Cauchy-Riemann operator $\xi \mapsto (\widetilde{\nabla}\xi)^{0,1}$, which defines a holomorphic structure on the bundle u^*TM. The index is the Euler characteristic of the Dolbeault cohomology of this bundle, given by the Riemann-Roch formula.

We next outline an argument which proves that $\mathcal{M}^*(A,\Sigma;J)$ is a manifold of the "correct dimension"

$$\dim(\mathcal{M}^*(A,\Sigma;J)) = n(2-2g) + 2c_1(u^*TM)$$

for a generic almost complex structure J. For this to hold, J must be allowed to vary in some space $\mathcal{J}$ of almost complex structures on M which is sufficiently large for the transversality argument to work. For example, $\mathcal{J}$ could be any subset of the space of all smooth structures which is open in the $\mathscr{C}^\infty$ topology; there is no need for M to be symplectic. The most important cases for us would be where M has a symplectic form ω and $\mathcal{J}$ is either the space of all ω-compatible or the space of all ω-tame almost complex structures. In all cases $\mathcal{J}$ carries the usual $\mathscr{C}^\infty$-topology. We write $\mathcal{J}^\ell$ for the corresponding space of $\mathscr{C}^\ell$ almost complex structures.

The basic observation is that the **universal moduli space** of simple curves

$$\mathcal{M}^*(A,\Sigma;J^\ell) := \{(u,J) \mid J \in \mathcal{J}^\ell, u \in \mathcal{M}^*(A,\Sigma;J)\}$$

is a separable Banach manifold when ℓ is sufficiently large (see Proposition 2.2.1). Moreover, the projection

$$\pi : \mathcal{M}^*(A,\Sigma;\mathcal{J}^\ell) \to \mathcal{J}^\ell$$

is a Fredholm map because its differential at a point (u,J) is essentially the operator D_u. In particular, π has the same index as D_u and is surjective at (u,J) precisely when D_u is onto. Because we are in a Banach manifold setting, we may apply the implicit function theorem to conclude that $\mathcal{M}^*(A,\Sigma;J)$ is indeed a finite dimensional manifold whose tangent space at u is the kernel of D_u whenever J is a regular value of π. Moreover, by the Sard-Smale theorem, the set of regular values of π is of second category in $\mathcal{J}^\ell$ for ℓ sufficiently large. Hence $\mathcal{M}^*(A,\Sigma;J)$ is a manifold of the right dimension for a generic element of $\mathcal{J}^\ell$.

An argument of Taubes allows us to show that this statement remains true for generic elements of the space $\mathcal{J}$ of *smooth* almost complex structures. It is based on the observation that the space of all simple J-holomorphic curves is a countable union of compact sets and for each of these compact sets the corresponding set of regular J is open. Hence the set of all regular J is a countable intersection of open sets $\mathcal{J}_{\mathrm{reg},K}$ and the substance of the matter is to show that these sets are also dense.

The above statements are proved in the next section. Here we sum up the main conclusions.

Definition 2.1.1. Fix a compact Riemann surface $(\Sigma, j_\Sigma, \mathrm{d}\,\mathrm{vol}_\Sigma)$ and a homology class $A \in H_2(M;\mathbb{Z})$. An almost complex structure J on M is called **regular (for A and Σ)** if D_u is onto for every $u \in \mathcal{M}^*(A,\Sigma;J)$. Given $\mathcal{J}$ as above, we denote by $\mathcal{J}_{\mathrm{reg}}(A,\Sigma)$ the set of all $J \in \mathcal{J}$ that are regular for A and Σ. In the case $\Sigma = S^2$ we shall abbreviate $\mathcal{J}_{\mathrm{reg}}(A) := \mathcal{J}_{\mathrm{reg}}(A,S^2)$.

Theorem 2.1.2 (Moduli space)**.** Assume $\mathcal{J} = \mathcal{J}(M,\omega)$ or $\mathcal{J} = \mathcal{J}_\tau(M,\omega)$. Fix a compact Riemann surface $(\Sigma, j_\Sigma, \mathrm{d}\,\mathrm{vol}_\Sigma)$ and a homology class $A \in H_2(M;\mathbb{Z})$.

1. If $J \in \mathcal{J}_{\mathrm{reg}}(A,\Sigma)$ then the space $\mathcal{M}^*(A,\Sigma;J)$ is a smooth manifold of dimension
$$\dim \mathcal{M}^*(A,\Sigma;J) = n(2-2g) + 2c_1(A).$$
It carries a natural orientation.

2. The set $\mathcal{J}_{\mathrm{reg}}(A,\Sigma)$ is of the second category in $\mathcal{J}$. This means that it contains an intersection of countably many open and dense subsets of $\mathcal{J}$.

The next task is to discuss the dependence of the manifolds $\mathcal{M}^*(A,\Sigma;J)$ on the choice of $J \in \mathcal{J}_{\mathrm{reg}}(A,\Sigma)$. A **(smooth) homotopy** of almost complex structures is a smooth map $[0,1] \to \mathcal{J}, \lambda \mapsto J_\lambda$. For any such homotopy define
$$\mathcal{M}^*(A,\Sigma;\{J_\lambda\}_\lambda) = ((\lambda,u) \mid 0 \le \lambda \le 1, u \in \mathcal{M}^*(A,\Sigma;J_\lambda)\}.$$
Given $J_0, J_1 \in \mathcal{J}_{\mathrm{reg}}(A,\Sigma)$ denote by $\mathcal{J}(J_0,J_1)$ the space of all smooth homotopies of almost complex structures connecting J_0 to J_1. In general, even if $\mathcal{J}$ is path connected, there does not exist a homotopy such that $J_\lambda \in \mathcal{J}_{\mathrm{reg}}(A,\Sigma)$ for every λ. In other words the space $\mathcal{M}^*(A,\Sigma;J_\lambda)$ may fail to be a manifold for some values of λ. However, we shall see that there is always a smooth homotopy such the space $\mathcal{M}^*(A,\Sigma;\{J_\lambda\}_\lambda)$ is a manifold.

Definition 2.1.3. Fix a compact Riemann surface $(\Sigma, j_\Sigma, \mathrm{d}\,\mathrm{vol}_\Sigma)$ and a homology class $A \in H_2(M;\mathbb{Z})$. Let $J_0, J_1 \in \mathcal{J}_{\mathrm{reg}}(A,\Sigma)$. A homotopy $[0,1] \to \mathcal{J}, \lambda \mapsto J_\lambda$ from J_0 to J_1 is called **regular (for A and Σ)** if
$$\Omega^{0,1}(\Sigma, u^*TM) = \mathrm{im}\, D_{J_{\lambda,u}} + \mathbb{R}v_\lambda$$
for every $(\lambda,u) \in \mathcal{M}^*(A,\Sigma;\{J_\lambda\}_\lambda)$ where
$$v_\lambda = (\partial_\lambda J_\lambda)\,\mathrm{d}u \circ j_\Sigma$$
is the image in $\Omega^{0,1}(\Sigma,u^*TM)$ of the tangent vector to the path $\lambda \mapsto J_\lambda$. The space of regular homotopies will be denoted by $\mathcal{J}_{\mathrm{reg}}(A,\Sigma;J_0,J_1)$.

A homotopy $\lambda \mapsto J_\lambda$ is regular precisely when it is transverse to the projection π. Intuitively speaking, one can think of the space $\mathcal{J}_{\mathrm{reg}}(A,\Sigma)$ of regular almost complex structures as the complement of a subvariety $\mathcal{S}$ of codimension 1 in the space $\mathcal{J}$. A smooth homotopy $\lambda \mapsto J_\lambda$ is regular if it is transversal to $\mathcal{S}$.

Theorem 2.1.4 (Parametric moduli space)**.** Assume $\mathcal{J} = \mathcal{J}(M,\omega)$ or $\mathcal{J} = \mathcal{J}_\tau(M,\omega)$. Fix a compact Riemann surface $(\Sigma, j_\Sigma, \mathrm{d}\,\mathrm{vol}_\Sigma)$ and a homology class $A \in H_2(M;\mathbb{Z})$. Let $J_0, J_1 \in \mathcal{J}_{\mathrm{reg}}(A,\Sigma)$.

1. If $\{J_\lambda\}_\lambda \in \mathcal{J}_{\mathrm{reg}}(A,\Sigma;J_0,J_1)$ then $\mathcal{M}^*(A,\Sigma;\{J_\lambda\}_\lambda)$ is a smooth oriented manifold with boundary
$$\partial\mathcal{M}^*(A,\Sigma;\{J_\lambda\}_\lambda) = \mathcal{M}^*(A,\Sigma;J_0) \cup \mathcal{M}^*(A,\Sigma;J_1).$$
The boundary orientation agrees with the orientation of $\mathcal{M}^*(A,\Sigma;J_1)$ and is opposite the orientation of $\mathcal{M}^*(A,\Sigma;J_0)$.

2. The set $\mathcal{J}_{\mathrm{reg}}(A,\Sigma;J_0,J_1)$ is of the second category in the space of all smooth homotopies $[0,1] \to \mathcal{J}, \lambda \mapsto J_\lambda$ from J_0 to J_1.

The above theorem shows that the moduli spaces $\mathcal{M}^*(A,\Sigma;J_0)$ and $\mathcal{M}^*(A,\Sigma;J_1)$ are oriented cobordant. However, until we establish some version of compactness this does not mean very much.

Elliptic regularity. It will be important to consider almost complex structures of class $\mathscr{C}^\ell$, rather than smooth ones, in order to obtain a parameter space with a Banach manifold structure. Hence we shall consider the space $\mathcal{J}^\ell = \mathcal{J}^\ell_\tau(M,\omega)$ of all ω-tame almost complex structures of class $\mathscr{C}^\ell$.

For a real number $p > 2$ and an integer $k \geq 1$ we shall denote by

$$\mathcal{B}^{k,p} := \{u \in \mathcal{W}^{k,p}(\Sigma, M) \mid [u] = A\}$$

the space of continuous maps $\Sigma \to M$ whose k-th derivatives are of class $\mathcal{L}^p$ and which represent the class $A \in H_2(M, \mathbb{Z})$. As explained in Section B.1, the space $\mathcal{W}^{k,p}(\Sigma, M)$ is the completion of $\mathscr{C}^\infty(\Sigma, M)$ with respect to a distance function based on the Sobolev $\mathcal{W}^{k,p}$-norm. This norm is defined as the sum of the $\mathcal{L}^p$ norms of all derivatives up to order k. The corresponding metric on $\mathscr{C}^\infty(\Sigma, M)$ can be defined by embedding M into some Euclidean space $\mathbb{R}^N$ and then using the a Riemannian metric on the ambient space $\mathcal{W}^{k,p}(\Sigma, \mathbb{R}^N)$. Alternatively, one can use a Riemannian metric on M and covariant derivatives to define the $\mathcal{W}^{k,p}$-norm on the tangent space $T_u\mathscr{C}^\infty(\Sigma, M) = \Omega^0(\Sigma, u^*TM)$ and then minimize the length of paths in $\mathscr{C}^\infty(\Sigma, M)$ with fixed endpoints. Since Σ and M are compact any two such $\mathcal{W}^{k,p}$ distance functions are compatible, and so the resulting completion does not depend on the choices. The space $\mathcal{W}^{k,p}(\Sigma, M)$ is a smooth separable Banach manifold with tangent space

$$T_u\mathcal{W}^{k,p}(\Sigma, M) = \mathcal{W}^{k,p}(\Sigma, u^*TM).$$

Local coordinate charts can be defined by using the exponential map along *smooth maps* $u : \Sigma \to M$ (see proof in next section). The space $\mathcal{B}^{k,p}$ is a component of $\mathcal{W}^{k,p}(\Sigma, M)$ and hence is also a smooth separable Banach manifold.

Remark 2.1.5. In order for the space $\mathcal{B}^{k,p}$ to be well defined we must assume that $kp > 2$. There are various reasons for this. Firstly, the very definition $\mathcal{B}^{k,p}$, in terms of local coordinate representations of class $\mathcal{W}^{k,p}$, requires this assumption. The point is this: the $\mathcal{W}^{k,p}$-norm is well-defined for maps between open sets in Euclidean space, but for a general manifold one needs a space which is invariant under composition with coordinate charts. Now the composition of a $\mathcal{W}^{k,p}$-map $u : \mathbb{R}^2 \to \mathbb{R}^{2n}$ with a $\mathscr{C}^k$-map in the target is again of class $\mathcal{W}^{k,p}$ precisely when $kp > 2$. Secondly, we will often use the Sobolev embedding theorem which says that under this condition by Rellich's theorem, this embedding is compact. Thirdly, the condition, $kp > 2$ is required to show that the product of two $\mathcal{W}^{k,p}$ functions is again of this class. In other words, the condition $kp > 2$ is needed to deal with the non-linearities.

The first key observation is that when J is smooth every J-holomorphic curve of class $\mathcal{W}^{1,p}$ with $p > 2$ is necessarily smooth. More precisely, we have the following regularity theorem which can be proved by ellliptic bootstrapping methods.

Proposition 2.1.6 (Elliptic regularity)**.** Assume J is an almost complex structure of class $\mathscr{C}^\ell$ with $\ell \geq 1$. If $u : \Sigma \to M$ is a J-holomorphic curve of class $\mathcal{W}^{1,p}$ with $p > 2$ then u is of class $\mathcal{W}^{\ell+1,p}$. In particular, u is of class $\mathscr{C}^\ell$, and if J is smooth, then so is u.

The above proposition shows that, for $J \in \mathcal{J}^\ell$, the moduli space of J-holomorphic curves of class $\mathcal{W}^{k,p}$ is independent of the choice of k so long as $k \leq \ell + 1$. When the equations are linearized we lose a derivative of J and so the condition that $k \leq \ell$ is needed for the operator D_u to be well defined on the appropriate Sobolev spaces.

The next result is a linear version of the above proposition. It relates the cokernel of the operator

$$D_u : \mathcal{W}^{k,p}(\Sigma, u^*TM) \to \mathcal{W}^{k-1,p}(\Sigma, \Lambda^{0,1} \otimes_J u^*TM) \tag{2.1}$$

to the kernel of the formal adjoint operator

$$D_u^* : \mathcal{W}^{k,p}(\Sigma, \Lambda^{0,1} \otimes_J u^*TM) \to \mathcal{W}^{k-1,p}(\Sigma, u^*TM)$$

and vice versa.

Proposition 2.1.7. Fix a positive integer ℓ and a constant $p > 2$. Let J be an ω-tame $\mathscr{C}^\ell$ almost complex structure on M and $u \in \mathcal{W}^{\ell,p}(\Sigma, M)$. Let $k \in \{1, \ldots, \ell\}$ and $q > 1$ such that $1/p + 1/q = 1$. Then the following holds.

1. The operators D_u and D_u^* are Fredholm with indices

$$\operatorname{ind} D_u = -\operatorname{ind} D_u^* = n(2 - 2g) + 2c_1(u^*TM),$$

   where g is the genus of Σ.

2. If $\eta \in L^q(\Sigma, \Lambda^{0,1} \times_J u^*TM)$ satisfies
$$\int_\Sigma \langle \eta, D_u\zeta \rangle \,\mathrm{d}\,\mathrm{vol}_\Sigma = 0$$
for all $\zeta \in \mathcal{W}^{k,p}(\Sigma, u * TM)$ then $\eta \in \mathcal{W}^{\ell,p}(\Sigma, \Lambda^{0,1} \otimes_J u^*TM)$ and $D_u^*\eta = 0$.
3. If $\zeta \in L^q(\Sigma, u^*TM)$ satisfies
$$\int_\Sigma \langle \zeta, D_u^*\eta \rangle \,\mathrm{d}\,\mathrm{vol}_\Sigma = 0$$
for all $\eta \in \mathcal{W}^{k,p}(\Sigma, \Lambda^{0,1} \otimes_J u^*TM)$ then $\zeta \in \mathcal{W}^{\ell,p}(\Sigma, u^*TM)$ and $D_u\zeta = 0$.

Proof. Assume first that u is smooth and consider the connection
$$\Omega^0(\Sigma, u^*TM) \to \Omega^1(\Sigma, u^*TM) : \zeta \to \nabla\zeta - \frac{1}{2} J(u)(\nabla_\zeta J)(u)\,\mathrm{d}u$$
on u^*TM. In general, this connection does not preserve the complex structure or the metric. The operator $D - u$ is the composition of this map with the projection $\Omega^1(\Sigma, u^*TM) \to \Omega^{0,1}(\Sigma, u^*TM)$, extended to the appropriate Sobolev completions. Hence D_u is a real linear Cauchy-Riemann operator of class $\mathcal{W}^{\ell-1,p}$ as in Definition C.1.2. Hence the result follows from Theorem C.2.3 with $q = p$ and $1/r + 1/p = 1$. □

The above proposition implies that the kernel and cokernel of D_u do not depend on the precise choice of the space on which the operator is defined. To see this, assume that J is of class $\mathscr{C}^\ell$ and that u is a J-holomorphic curve. Then, by elliptic regularity, u is of class $\mathscr{C}^\ell$. Now consider the operator D_u, in Eq. (2.1), with $k \le \ell$. By the above proposition, every element in the kernel of D_u is necessarily of class $\mathcal{W}^{\ell,p}$ for any p and so the kernel of D_u does not depend on the choice of k and p as long as $k \le \ell$. The same holds for the cokernel. In particular, the operator D_u is onto for some choice of k and p iff it is onto for all such choices.

2.2 Transversality

The proof of Theorem 2.1.2 is based on an infinite dimensional version of Sard's theorem which is due to Smale.

In order to get a result in the smooth category, our strategy is to prove that the set of regular almost complex structures of class $\mathscr{C}^\ell$ is generic (i.e., of second category) with respect to the $\mathscr{C}^\ell$ topology and then to take the intersection of these sets over all ℓ. This approach is due to Taubes. Throughout we shall denote by $\mathcal{J}^\ell$ either the space $\mathcal{J}^\ell(M,\omega)$ of all ω-compatible almost complex structures of class $\mathscr{C}^\ell$ or the space $\mathcal{J}^\ell_\tau(M,\omega)$ of all ω-tame almost complex structures of class $\mathscr{C}^\ell$.

The universal moduli space. The universal moduli space
$$\mathcal{M}^*(A,\Sigma;\mathcal{J}^\ell) := \{(u,J) \mid J \in \mathcal{J}^\ell, u \in \mathcal{M}^*(A,\Sigma;J)\}$$
consists of all simple J-holomorphic curves $u : \Sigma \to M$ representing the class A, where J varies over the space $\mathcal{J}^\ell$. Recall from Proposition 2.1.6 that every J-holomorphic curve is of class $\mathscr{C}^\ell$ whenever J is of class $\mathscr{C}^\ell$. Hence we may view $\mathcal{M}^*(A,\Sigma;\mathcal{J}^\ell)$ as a subset of $\mathcal{B}^{k,p} \times \mathcal{J}^\ell$ for any $p > 2$ and any $k \in \{1,\dots,\ell\}$, where $\mathcal{B}^{k,p}$ denotes the space of $\mathcal{W}^{k,p}$-maps $u : \Sigma \to M$ representing the class A. Care must be taken in the case $k = 1$, because $\mathcal{W}^{1,p}$ function is not necessarily continuously differentiable. A continuous function $u : \Sigma \to M$ is called **somewhere injective** if there is a point $z \in \Sigma$ and a constant $\delta > 0$ such that, for every $\zeta \in \Sigma$, we have
$$d_M(u(z), u(\zeta)) \ge \delta d_\Sigma(z,\zeta).$$
This condition is independent of the choice of the metrics on M and Σ used to express it. Note also that, if u is continuously differentiable, then this condition agrees with the notion of somewhere injectivity introduced earlier (Section 1.3.4).

The parameter space $\mathcal{J}^\ell$ is a smooth separable Banach manifold. Its tangent space $T_J\mathcal{J}^\ell$ at J (in the ω-compatible case) consists of $\mathscr{C}^\ell$-sections Y of the bundle $\mathrm{End}(TM, J, w)$ whose fiber at $x \in M$ is the space of linear maps $Y : T_xM \to T_xM$ such that
$$YJ + JY = 0, \qquad \omega(Yv, w) + \omega(v, Yw) = 0.$$

The first equation is the derivative of the identity $J^2 = -\mathbb{1}$, while the second comes from the compatibility condition. Equivalently, one could express these conditions as

$$Y = Y^* = JYJ$$

where Y^* denotes the adjoint operator with respect to the metric g_J. The space of $\mathscr{C}^\ell$-sections of the bundle $\mathrm{End}(TM, J, \omega) \to M$ is a Banach space and gives rise to a local model for the space $\mathcal{J}^\ell$ via $Y \mapsto J\exp(-JY)$. The corresponding section of $\mathscr{C}^\infty$-sections is not a Banach space but only a Frechet space. Our convention is that these spaces with no superscripts consist of elements what are $\mathscr{C}^\infty$-smooth.

Proposition 2.2.1. Fix a homology class $A \in H_2(M, \mathbb{Z})$, an integer $\ell \geq 2$, a real number $p > 2$, and an integer $k \in \{1, \dots, \ell\}$. Then the universal moduli space $\mathcal{M}^*(A, \Sigma; \mathcal{J}^\ell)$ is a separable $\mathscr{C}^{\ell-k}$ Banach submanifold of $\mathcal{B}^{k,p} \times \mathcal{J}^\ell$.

As we shall see this proposition holds because the tangent space $T_J\mathcal{J}$ is sufficiently large. In particular, the next lemma shows that it acts transitively on T_xM at each point $x \in M$.

Lemma 2.2.2. Let $\xi, \eta \in \mathbb{R}^{2n}$ be two nonzero vectors. Then there exists a matrix $Y \in \mathbb{R}^{2n\times 2n}$ such that

$$Y = Y^T = J_0 Y J_0, \qquad Y\xi = \eta.$$

Proof. An explicit formula for Y is

$$Y := \frac{1}{|\xi|^2}\left(\eta\xi^T + \xi\eta^T + J_0(\eta\xi^T + \xi\eta^T)J_0\right) - \frac{1}{|\xi|^4}\left(\langle\eta,\xi\rangle(\xi\xi^T + J_0\xi\xi^T J_0) - \langle\eta, J_0\xi\rangle(J_0\xi\xi^T - \xi\xi^T J_0)\right).$$

The formula is constructed in stages. The first term is a matrix that takes ξ to tη ,the second makes it symmetric while the third and fourth lead to the equation $Y = J_0YJ_0$. But the resulting matrix no longer takes ξ to η; the last four terms adjust for this. □

Proof (of Proposition 2.2.1). Consider the $\mathscr{C}^{\ell\text{-k}}$-Banach bundle

$$\mathcal{E}^{k-1,p} \to \mathcal{B}^{k,p} \times \mathcal{J}^\ell,$$

whose fiber over (u, J) is the space

$$\mathcal{E}^{k-1,p}_{(u,J)} = \Omega^{k-1,p}(\Sigma, \Lambda^{0,1} \otimes_J u^*TM)$$

of J-antilinear 1-forms on Σ of class $\mathcal{W}^{k-1,p}$ with values in the bundle u^*TM. The $\mathscr{C}^{\ell\text{-k}}$-Banach manifold structure of $\mathcal{E}^{k-1,p}$ can be constructed as follows. Fix any smooth metric on M to identify a neighborhood of zero in the Banach space $\Omega^{k,p}(\Sigma, u^*TM)$ via $\xi \mapsto \exp_u(\xi)$. This gives coordinate charts on $\mathcal{B}^{k,p}$ with smooth transition maps. Now trivialize the bundle $\mathcal{E}^{k-1,p}$ over such a coordinate chart by using the Hermitian connection $\widetilde{\nabla}$. Since J is of class $\mathscr{C}^\ell$, the connection $\widetilde{\nabla}$ and its parallel transport maps are of class $\mathscr{C}^{\ell-1}$. Hence, if we differentiate the transition maps arising from these trivializations $\ell - k$ times, the result is still well defined in $\mathcal{W}^{k-1,p}$. This constructs local trivializations over open sets $\mathcal{N}(u)$ in the slice $\mathcal{B}^{k,p} \times \{J\}$. To extend this over neighborhoods of the form $\mathcal{N}(u) \times \mathcal{N}(J)$ first trivialize over a suitable neighborhood $\{u\} \times \mathcal{N}(J)$ via the isomorphism

$$\Lambda^{0,1}_J \otimes u^*TM \to \Lambda^{0,1}_{J'} \otimes u^*TM : \quad \alpha \mapsto \frac{1}{2}(\alpha - J' \circ \alpha \circ j),$$

and then extend this trivialization over each slice $\mathcal{N}(u) \times \{J'\}$ using parallel translation as before. This shows that $\mathcal{E}^{k-1,p}$ is a Banach space bundle of class $\mathscr{C}^{\ell-k}$.

The map $(u, J) \mapsto \overline{\partial}_J(u)$ defines a $\mathscr{C}^{\ell-k}$ section of the bundle $\mathcal{E}^{k-1,p} \times \mathcal{B}^{k,p} \times \mathcal{J}^\ell$. Denote the section by

$$\mathcal{F} : \mathcal{B}^{k,p} \times \mathcal{J}^\ell \to \mathcal{E}^{k-1,p}, \qquad \mathcal{F}(u, J) = \overline{\partial}_J(u).$$

We must prove that the vertical differential

$$D\mathcal{F}(u, J) : \mathcal{W}^{k,p}(\Sigma, u^*TM) \times \mathscr{C}^\ell(M, \mathrm{End}(TM, J, \omega)) \to \Omega^{k-1}(\Sigma, \Lambda^{0,1} \otimes_J u^*TM)$$

is surjective whenever u is simple and $\mathcal{F}(u,J)=0$. One can show that (see [2], Proposition 3.1.1) this differential is given by the formula

$$D\mathcal{F}(u,J)(\xi,Y)=D_u\xi+\frac{1}{2}Y(u)\circ \mathrm{d}u\circ j.$$

Since D_u is Fredholm the operator $D\mathcal{F}(u,J)$ has a closed image and it suffices to prove that the image is dense whenever $\overline{\partial}_J(u)=0$.

We prove this first in the case $k=1$. If the image is not dense then, by Hahn-Banach theorem, there exists a nonzero section $\eta\in\mathcal{L}^q(\Lambda^{0,1}\otimes_J u^*TM)$, with $1/p+1/q=1$, which annihilates the image of $D\mathcal{F}(u,J)$. This means that

$$\int_\Sigma \langle\eta, D_U\xi\rangle\,\mathrm{d}\,\mathrm{vol}\,\Sigma=0 \tag{2.2}$$

for every $\xi\in\omega^{1,p}(\Sigma,u^*TM)$ and

$$\int_\Sigma \langle\eta, Y(u)\circ \mathrm{d}u\circ j\rangle\,\mathrm{d}\,\mathrm{vol}_\Sigma=0 \tag{2.3}$$

for every $Y\in\mathscr{C}^\ell(M,\mathrm{End}(TM,J,\omega))$, where $j:=j_\Sigma$. It follows from Eq. (2.2) and Proposition 2.1.7 that η is of class $\mathcal{W}^{1,p}$ and $D_u^*\eta=0$.

Since u is simple, Proposition 1.3.9 asserts that the set of injective points of u is open and dense in Σ. Let $z_0\in\Sigma$ be such an injective point, i.e.,

$$\mathrm{d}u(z_0)\neq 0,\qquad u^{-1}(u(z_0))=\{z_0\}.$$

We shall prove that η vanishes at z_0. Assume, by contradiction, that $\eta(z_0)\neq 0$. Then, since $\mathrm{d}u(z_0)\neq 0$, it follows from Lemma 2.2.2 that there exists an endomorphism $Y_0\in\mathrm{End}(T_{u(z_0)}M,J_{u(z_0)},\omega_{u(z_0)})$ such that such that

$$\langle\eta(z_0),Y_0\circ \mathrm{d}u(z_0)\circ j(z_0)\rangle>0.$$

Now choose any section $Y\in\mathscr{C}^\ell(M,\mathrm{End}(TM,J,\omega))$ such that $Y(u(z_0))=Y_0$. Then the scalar function $\langle\eta,Y(u)\circ \mathrm{d}u\circ j\rangle$ on Σ is positive in some open neighborhood $V_0\in\Sigma$ of z_0. Since z_0 is an injective point of u the compact set $u(\Sigma\setminus V_0)$ does not contain the point $u(z_0)$. Hence there exists an open neighborhood $U_0\subseteq M$ of $u(z_0)$ such that $u(\Sigma\setminus V_0)\cap U_0=\emptyset$ and hence

$$u^{-1}(U_0)\subseteq V_0.$$

Now choose a smooth cutoff function $\beta: M\to[0,1]$ supported in U_0 such that $\beta(u(z_0))=1$. Then the function $\beta(u)\langle\eta,Y(u)\circ \mathrm{d}u\circ j\rangle$ on Σ is supported in V_0 is nonnegative, and is positive somewhere, hence the integral on the left hand side of Eq. (2.3), with Y replaced by βY, does not vanish. This contradiction shows that $\eta(z_0)=0$. Since this holds for every injective point z_0 of u, and the set of injective points is dense in Σ, it follows that η vanishes almost everywhere. Since η is continuous we obtain $\eta\equiv 0$. This we have proved that $D\mathcal{F}(u,J)$ has a dense image and is therefore onto in the case $k=1$.

To prove surjectivity for general k, let $\eta\in\Omega^{k-1,p}(\Sigma,\Lambda^{0,1}\otimes_J u^*TM)$ be given. Then, by surjectivity for $k=1$, there exists a pair

$$(\xi,Y)\in\Omega^{1,p}(\Sigma,u^*TM)\times\mathscr{C}^\ell(M,\mathrm{End}(TM,J,\omega))$$

such that

$$D\mathcal{F}(u,J)(\xi,Y)=\eta.$$

Now the above formula for $D\mathcal{F}(u,J)$ shows that

$$D_u\xi=\eta-\frac{1}{2}Y(u)\circ \mathrm{d}u\circ j\in\mathcal{W}^{k-1,p}$$

and, by elliptic regularity, $\xi\in\mathcal{W}^{k,p}(\Sigma,u^*TM)$. Hence $D\mathcal{F}(u,J)$ is onto for every pair $(u,J)\in\mathcal{M}^*(A,\Sigma;\mathcal{J}^\ell)$. One can show that because D_u is a Fredholm operator, it has a right inverse. Hence it follows from the infinite dimensional implicit function theorem (Theorem A.3.3) that the space $\mathcal{M}^*(A,\Sigma;\mathcal{J}^\ell)$ is a $\mathscr{C}^{\ell-k}$ Banach submanifold of $\mathcal{B}^{k,p}\times\mathcal{J}^\ell$. Since $\mathcal{B}^{k,p}\times\mathcal{J}^\ell$ is separable so is $\mathcal{M}^*(A,\Sigma;\mathcal{J}^\ell)$. □

Remark 2.2.3. The assertion of Proposition 2.2.1 continues to hold for more general sets $\mathcal{J}$ of almost complex structures. For example, it suffices to consider the space of ω-compatible almost complex structures that agree with a given almost complex structure J_0 on the complement of an open set $U \subseteq M$, provided that every J-holomorphic curve $u : \Sigma \to M$ representing the class A passes through the set U (for every J). However, in this case, the proof requires Aronszajn's theorem. Namely, if $D_u^* \eta = 0$, then

$$0 = D_u D_u^* \eta = \Delta \eta + \text{ lower order terms}$$

and it follows from Aronszajn's theorem, that if η vanishes on some open set then $\eta \equiv 0$. Now it suffices to choose infinitesimal almost complex structures Y with support in the set U to guarantee that the annihilator in the above proof vanishes on some open set.

Proof (of Theorem 2.1.2 (1)). Suppose given an almost complex structure $J \in \mathcal{J}_{\mathrm{reg}}(A, \Sigma)$ and a J-holomorphic curve $u \in \mathcal{M}^*(A, \Sigma; J)$. By Proposition 2.1.6, u is smooth. Given $\xi \in \Omega^0(\Sigma, u^*TM)$, let

$$\Phi_u(\xi) : u^*TM \to \exp_u(\xi)^*TM$$

denote the complex bundle isomorphism, given by parallel transport along the geodesics $s \mapsto \exp_{u(x)}(s\xi(z))$ with respect to the complex linear connection $\widetilde{\nabla}_v X := \nabla_v X - 1/2\, J(\nabla_v J)X$ (induced by the Levi-Civita connection ∇ of the Riemannian metric coming from J!). A map

$$\mathcal{F}_u : \Omega^0(\Sigma, u^*TM) \to \Omega^{0,1}(\Sigma, u^*TM)$$

can be defined by

$$\mathcal{F}_u(\xi) := \Phi_u(\xi)^{-1} \overline{\partial}_J(\exp_u(\xi)),$$

that is, taking $\overline{\partial}$ and then parallel transporting back to the same tangent space. Given an integer $k \geq 1$ and a real number $p > 2$ consider the map

$$\mathcal{F}_u : \mathcal{W}^{k,p}(\Sigma, u^*TM) \to \mathcal{W}^{k-1,p}(\Sigma, \Lambda^{0,1} \otimes_J u^*TM)$$

defined similarly. Then a $\mathcal{W}^{k,p}$-neighborhood of zero in $\mathcal{F}_u^{-1}(0)$ is diffeomorphic to a $\mathcal{W}^{k,p}$-neighborhood of u in $\mathcal{M}^*(A, \Sigma; J)$ via the map $\xi \mapsto \exp_u(\xi)$. One can give an explicit formula for the differential of $\mathcal{F}_u$ at zero, that is, $D_u : \Omega^0(\Sigma, u^*TM) \to \Omega^{0,1}(\Sigma, u^*TM)$, $D_u \xi := \mathrm{d}\mathcal{F}_u(0)\xi$

$$D_u \xi = \frac{1}{2}(\nabla \xi + J(u) \nabla \xi \circ j_\Sigma) - \frac{1}{2} J(u)(\nabla_\xi J)(u) \partial_J(u).$$

Assuming the above and from the definition of $\mathcal{J}_{\mathrm{reg}}(A, \Sigma)$, the differential $\mathrm{d}\mathcal{F}_u(0) = D_u$ is surjective. Since $\mathcal{F}_u$ is a smooth map between Banach spaces, it follows from Theorem A.3.3 that $\mathcal{F}_u^{-1}(0)$ intersects a sufficiently small neighborhood of zero in a smooth submanifold of dimension $n(2-2g) + 2c_1(A)$. The image of this submanifold under the map $\xi \mapsto \exp_u(\xi)$ is a smooth submanifold of $\mathcal{W}^{k,p}(\Sigma, M)$ and agrees with a neighborhood of u in $\mathcal{M}^*(A, \Sigma; J)$. Hence $\mathcal{M}^*(A, \Sigma; J)$ is a smooth submanifold of $\mathcal{W}^{k,p}(\Sigma, M)$. We emphasize that the coordinate charts on $\mathcal{M}^*(A, \Sigma; J)$ obtained in this way are independent of k and p, because of all elements in $\mathcal{F}_u^{-1}(0)$ near zero are necessarily smooth (Proposition 2.1.6).

To complete the proof of the moduli space (1), we need to show that the moduli space has a canonical orientation. To understand why, observe first that the tangent space $T_u\mathcal{M}^*(A, \Sigma; J)$ is the kernel of D_u. One can show that, the operator D_u is the sum of the operator $\xi \mapsto (\widehat{\nabla}\xi)^{0,1}$ and an operator of the form $\xi \mapsto T(\xi, \partial_J(u))$, where the first of these has order one and commutes with J, while the second has order zero and anti commutes with J. Hence the kernel of D_u will in general not be invariant under J and so J might not determine an almost complex structure on $T_u\mathcal{M}^*(A, \Sigma; J)$. However, by multiplying the second part of D_u by a constant which tends to zero, one can homotop D_u through Fredholm operators D_u^t to a Fredholm operator which does commute with J, namely the complex linear Cauchy-Riemann operator $\xi \mapsto (\widehat{\nabla}\xi)^{0,1}$.

Now one can show that the determinant

$$\det(D) = \Lambda^{\max}(\ker D) \otimes \Lambda^{\max}(\ker D^*)$$

of a Fredholm operator $D : X \to Y$ between complex Banach spaces carries a natural orientation whenever the operator D is complex linear (see [2], Section A.2). In our case the complex antilinear part of the operator D_u is compact and hence the determinant line $\det(D_u) = \Lambda^{\max}(\ker D_u)$ inherits a natural orientation from the complex linear part of D_u. These orientations of $\det(D_u)$ for $u \in \mathcal{M}^*(A, \Sigma; J)$ determine the natural orientation on $\mathcal{M}^*(A, \Sigma; J)$. □

Remark 2.2.4. Note that if J is integrable, then D_u commutes with J, and so J induces an (integrable) almost complex structure on $\mathcal{M}^*(A, \Sigma; J)$. By definition, this complex structure is compatible with the orientation described in the above proof. In particular, if the index is zero and J is regular and integrable, every element of the moduli space $\mathcal{M}^*(A, \Sigma; J)$ gives the contribution $+1$ to the count of J-holomorphic A curves. Hence, in this situation when this number is negative, one cannot find regular integrable J.

Proof (of Theorem 2.1.2 (2)). We now must show that the set $\mathcal{J}_{\mathrm{reg}}(A, \Sigma)$ of smooth regular almost complex structures J is of the second category in $\mathcal{J}$. The argument is based on the properties of the projection

$$\pi : \mathcal{M}^*(A, \Sigma; \mathcal{J}^\ell) \to \mathcal{J}^\ell.$$

By Proposition 2.2.1 with $k = 1$, this is a $\mathscr{C}^{\ell-1}$-map between separable $\mathscr{C}^{\ell-1}$ Banach manifolds. The tangent space

$$T_{(u,J)}\mathcal{M}^*(A, \Sigma; \mathcal{J}^\ell) \subseteq \Omega^{1,p}(\Sigma, u^*TM) \times \mathscr{C}^\ell(M, \mathrm{End}(TM, J, \omega))$$

consists of all pairs (ξ, Y) such that

$$D_u\xi + \frac{1}{2}Y(u) \circ du \circ j_\Sigma = 0.$$

Moreover, the derivative

$$d\pi(u, J) : T_{(u,J)}\mathcal{M}^*(A, \Sigma; \mathcal{J}^\ell) \to T_J\mathcal{J}^\ell$$

is the projection $(\xi, Y) \mapsto Y$. Hence the kernel of $d\pi(u, J)$ is isomorphic to the kernel of D_u. Again, one can show that its cokernel also is isomorphic to the cokernel of D_u. It follows that $d\pi(u, J)$ is a Fredholm operator with the same index as D_u. Moreover, the operator $d\pi(u, J)$ is onto precisely when D_u is onto. Hence a regular value J of π is an almost complex structure with the property that D_u is onto for every simple J-holomorphic curve $u \in \mathcal{M}^*(A, \Sigma; J) = \pi^{-1}(J)$. In other words, the set

$$\mathcal{J}^\ell_{\mathrm{reg}}(A, \Sigma) := \{J \in \mathcal{J}^\ell \mid D_u \text{ is onto for all } u \in \mathcal{M}^*(A, \Sigma; J)\}$$

of regular almost complex structures is precisely the set of regular values of π. By the Sard Smale theorem (Theorem A.4.3), this set is co-meager in the sense of Baire (a countable intersection of open and dense sets). here we use the fact that the manifold $\mathcal{M}^*(A, \Sigma; \mathcal{J}^\ell)$ and the projection π are of class $\mathscr{C}^{\ell-1}$. Hence we can apply the Sard-Smale theorem whenever

$$\ell - 2 \geq \mathrm{index}\,\pi = \mathrm{index}\,D_u = n(2 - 2g) + 2c_1(A).$$

Thus we have proved that the set $\mathcal{J}^\ell_{\mathrm{reg}}$ is dense in $\mathcal{J}^\ell$ with respect to the $\mathscr{C}^\ell$-topology for ℓ sufficiently large.

We shall now use an argument due to Taubes, to deduce that $\mathcal{J}_{\mathrm{reg}}(A, \Sigma)$ is co-meager in $\mathcal{J}$ with respect to the $\mathscr{C}^\infty$-topology.

Consider the set

$$\mathcal{J}_{\mathrm{reg},K}(A, \Sigma) \subseteq \mathcal{J}(A, \Sigma)$$

of all smooth almost complex structures $J \in \mathcal{J}$ such that the operator D_u is onto for every J-holomorphic curve $u : \Sigma \to M$ representing the class A which satisfies

$$\|du\|_{\mathcal{L}^\infty} \leq K \tag{2.4}$$

and for which there exists a point $z \in \Sigma$ such that

$$\inf_{\zeta \neq z} \frac{d(u(z), u(\zeta))}{d(z, \zeta)} \geq \frac{1}{K}. \tag{2.5}$$

The latter condition guarantees that u is simple. Moreover, every simple J-holomorphic curve $u : \Sigma \to M$ satisfies these two conditions for some value of $K > 0$ and some point $z \in \Sigma$. Hence

$$\mathcal{J}_{\text{reg}}(A, \Sigma) = \cap_{K>0}\mathcal{J}_{\text{reg},K}(A, \Sigma).$$

We shall prove that each set $\mathcal{J}_{\text{reg},K}(A, \Sigma)$ is open and dense in $\mathcal{J}$ with respect to the $\mathscr{C}^\infty$-topology. We first prove that $\mathcal{J}_{\text{reg},K}(A, \Sigma)$ is open or equivalently, that its complement is closed. hence assume that the sequence $J_\nu \notin \mathcal{J}_{\text{reg},K}(A, \Sigma)$ converges to $J \in \mathcal{J}$ in the $\mathscr{C}^\infty$-topology. Then there exist, for every ν, a point $z_\nu \in \Sigma$ and a J_ν-holomorphic curve $u_\nu \in \mathcal{M}^*(A, \Sigma; J_\nu)$ which satisfies Eq. (2.4) and Eq. (2.5), with z replaced by z_ν, such that the operator D_{u_ν} is not surjective. It follows from standard elliptic bootstrapping arguments that there exists a subsequence u_ν, which converges, uniformly with all derivatives, to a smooth J-holomorphic curve $u : \Sigma \to M$ (See Theorem B.2.6). Choose the subsequence such that z_{ν_i} converges to z. Then the limit curve u satisfies Eq. (2.4) and Eq. (2.5) for this point z and, moreover, since the operators $D_{u_{\nu_i}}$ are not surjective, it follows that D_u cannot be surjective either. This shows that $J \notin \mathcal{J}_{\text{reg},K}(A, \Sigma)$ and thus we have proved that the complement of $\mathcal{J}_{\text{reg},K}(A, \Sigma)$ is closed in the $\mathscr{C}^\infty$-topology.

Next, we prove that the set $\mathcal{J}_{\text{reg},K}(A, \Sigma)$ is dense in $\mathcal{J}$ with respect to the $\mathscr{C}^\infty$-topology. To see this note first that

$$\mathcal{J}_{\text{reg},K}(A, \Sigma) = \mathcal{J}^\ell_{\text{reg},K}(A, \Sigma) \cap \mathcal{J}$$

where $\mathcal{J}^\ell_{\text{reg},K}(A, \Sigma) \subseteq \mathcal{J}^\ell$ is the set of $J \in \mathcal{J}^\ell(M, \omega)$ such that the operator D_u is onto for every J-holomorphic curve $u : \Sigma \to M$ of class $\mathscr{C}^\ell$ that represents the class A and satisfies Eq. (2.4) and Eq. (2.5) for some $z \in \Sigma$. The same argument as above shows that $\mathcal{J}^\ell_{\text{reg},K}(A, \Sigma)$ is open in $\mathcal{J}^\ell$ with respect to the $\mathscr{C}^\ell$ topology.

Now let $J \in \mathcal{J}$. Since $\mathcal{J}^\ell_{\text{reg}}(A, \Sigma)$ is dense in $\mathcal{J}^\ell$ for large ℓ there exists a sequence $J_\ell \in \mathcal{J}^\ell_{\text{reg}}(A, \Sigma)$, $\ell \geq \ell_0$, such that

$$\|J' - J_\ell\|_{\mathscr{C}^\ell} \leq 2^{-\ell}.$$

Since $J_\ell \in \mathcal{J}^\ell_{\text{reg},K}(A, \Sigma)$ and $\mathcal{J}^\ell_{\text{reg},K}(A, \Sigma)$ is open in the $\mathscr{C}^\ell$-topology there exists an $\varepsilon_\ell > 0$ such that, for every $J' \in \mathcal{J}^\ell$,

$$\|J' - J_\ell\| < \varepsilon_\ell \qquad \Longrightarrow \qquad J' \in \mathcal{J}^\ell_{\text{reg},K}(A, \Sigma).$$

Choose $J'_\ell \in \mathcal{J}$ to be any smooth element such that

$$\|J'_\ell - J_\ell\|_{\mathscr{C}^\ell} \leq \min\{\varepsilon_\ell, 2^{-\ell}\}.$$

Then

$$J'_\ell \in \mathcal{J}^\ell_{\text{reg},K}(A, \Sigma) \cap \mathcal{J} = \mathcal{J}_{\text{reg},K}(A, \Sigma)$$

and the sequence J'_ℓ converges to J in the $\mathscr{C}^\infty$-topology. This shows that the set $\mathcal{J}_{\text{reg},K}(A, \Sigma)$ is dense in $\mathcal{J}$ as claimed. Thus $\mathcal{J}_{reg}(A, \Sigma)$ is the intersection of the countable number of open sets $\mathcal{J}_{\text{reg},K}(A, \Sigma)$, $K \in \mathbb{N}$, and is so of the second category as required. □

Proof (of Theorem 2.1.4). The proof is almost word by word the same as that of the above. Let $\mathcal{J}^\ell(J_0, J_1)$ denote the space $\mathscr{C}^\ell$-homotopies $[0, 1] \to \mathcal{J}^\ell(M, \omega)$, $\lambda \mapsto J_\lambda$ from J_0 to J_1. One then considers the universal moduli space $\mathcal{M}^*(A, \Sigma; \mathcal{J}^\ell)$ is a $\mathscr{C}^{\ell-1}$ Banach manifold and that the projection onto the space $\mathcal{J}^\ell(J_0, J_1)$ is a $\mathscr{C}^{\ell-1}$ Fredholm map. The regular values of this map are the required regular homotopies and, by the Sard-Smale theorem, they form a dense subset of $\mathcal{J}^\ell(J_0, J_1)$ for ℓ sufficiently large. The reduction of the smooth case to the $\mathscr{C}^\ell$-case is as in the earlier proof. □

Remark 2.2.5. The proof of Theorem 2.1.2, shows that a point (u, J) in the (smooth) universal moduli space $\mathcal{M}^*(A, \Sigma; \mathcal{J})$ is regular in the sense of D_u being onto iff the projection $d\pi(u, J) : T_{(u,J)}\mathcal{M}^*(A, \Sigma; \mathcal{J}) \to T_J\mathcal{J}$ is surjective. By the implicit function theorem, this implies that any smooth path $[0, 1] \to \mathcal{J}, t \mapsto J_t$ which starts at $J_0 = J$ can be lifted, on some interval $[0, \varepsilon)$, to a path $[0, \varepsilon) \to \mathcal{M}^*(A, \Sigma; \mathcal{J}), t \mapsto (u_t, J_t)$ in the universal moduli space which starts at $u_0 = u$. Thus any regular curve (u, J) persists when J is perturbed. In contrast, if (u, J) is not regular there might be no nearby curve when J is perturbed.

2.3 A regularity criterion

One of the key ingredients in the proof of Gromov's non-squeezing theorem is the application of a regularity criterion. We shall be interested in $\Sigma = S^2 = \mathbb{C}P^1$. Grothendieck proved that any holomorphic vector bundle E over $\mathbb{C}P^1$ is holomorphically equivalent to a direct sum of holomorphic line bundles. Moreover, this splitting is unique up to the order of the summands. Hence,

$$E = L_1 \oplus \cdots \oplus L_n$$

is completely characterized by the set of Chern numbers $c_1(L_1), \ldots, c_1(L_n)$. In particular, Grothendieck's theorem applies to the vector bundle $E = u^*TM$, which is holomorphic whenever $u : S^2 \to M$ is a J-holomorphic sphere for an (integrable) complex structure J. By the Whitney sum formula, $c_1(E) = \sum_i c_1(L_i)$ is a topological invariant, but the numbers $c_1(L_1), \ldots, c_1(L_n)$ in the decomposition of $E = u^*TM$ may vary as $u : S^2 \to M$ varies in a connected component of the space of J-holomorphic spheres.

The next lemma gives a criterion for regularity, and we outline a proof here based on observations from complex algebraic geometry.

Lemma 2.3.1. Assume J is integrable and let $u : \mathbb{C}P^1 \to M$ be a J-holomorphic sphere. Suppose that every summand of u^*TM has Chern number $c_1 \geq -1$. Then D_u is onto.

Proof. If J is integrable, then the operator D_u is complex linear and is exactly the Dolbeault derivative $\overline{\partial}$ determined by the holomorphic coordinate charts of M. Hence, it respects the splitting of u^*TM into holomorphic line bundles, and we can consider each line sub-bundle L of (TM, J) separately. Further, the cokernel of $D_u = \overline{\partial} : \Omega^0(\mathbb{C}P^1, L) \to \Omega^{0,1}(\mathbb{C}P^1, L)$ is precisely the Dolbeault cohomology group $H^{0,2}_{\overline{\partial}}(\mathbb{C}P^2, L)$. Therefore, it suffices to show this group vanishes whenever $c_1(L) \geq -1$. Now, for any holomorphic line bundle $L \to \mathbb{C}P^1$,

$$H^{0,1}_{\overline{\partial}}(\mathbb{C}P^1, L) \simeq \left(H^{1,0}(\mathbb{C}P^1, L^*)\right)^*.$$

Here, $H^{1,0}_{\overline{\partial}}(\mathbb{C}P^1, L^*)$ is the space of holomorphic 1-forms with values in the dual bundle L^* and so is isomorphic to the space $H^0(\mathbb{C}P^1, L^* \otimes K)$ of holomorphic sections of the bundle $L^* \otimes K$ where $K := T^*\mathbb{C}P^1$ is the canonical bundle. This is a special case of Kodaira-Serre duality which can be checked directly by considering the the transition maps. Now a line bundle L' over $\mathbb{C}P^1$ has nonzero holomorphic sections iff $c_1(L') \geq 0$. This is a special case of the Kodaira vanishing theorem, and holds by positivity of intersections and the interpretation of $c_1(L')$ as the self-intersection number of the zero section. Thus D_u is surjective iff $c_1(L^* \otimes K) < 0$ for every holomorphic summand L of E. Since $c_1(L^* \otimes K) = -c_1(L) - 2$, this is equivalent to $c_1(L) \geq -1$. □

The previous argument generalizes to almost complex manifolds as follows.

Lemma 2.3.2. Let $E \to S^2$ be a complex vector bundle of rank n and

$$D : \Omega^0(S^2, E) \to \Omega^{0,1}(S^2, E)$$

be a real linear CR operator (defined in Section C.1). Suppose that there exists a splitting $E = L_1 \oplus \cdots \oplus L_n$ into complex line bundles such that each subbundle $L_1 \oplus \cdots \oplus L_k$, $k = 1, \ldots, n$, is invariant under D. Then D is surjective iff $c_1(L_k) \geq -1$ for every k.

Proof. Let $\pi_k : E \to L_k$ denote the projection onto the k-th summand and define the operator $D_{kj} : \Omega^0(S^2, L_j) \to \Omega^{0,1}(S^2, L_k)$ by

$$D_{kj}\xi_j := \pi_k(D\xi_j)$$

for $\xi_j \in \Omega^0(S^2, L_j) \subseteq \Omega^0(S^2, E)$. Then $D_{kj} = 0$ for $k > j$, D_{kk} is a real linear Cauchy-Riemann operator on L_k, and D_{kj} is a zeroth order operator for $k < j$. Now the equation $D\xi = \eta$, for $\xi \in \Omega^0(S^2, E)$ and $\eta \in \Omega^{0,1}(S^2, E)$, can be written in the form

$$\eta_k = \sum_{j=k}^{n} D_{kj}\xi_j, \qquad k = 1, \ldots, n,$$

where $\xi_j = \pi_j(\eta)$ and $\eta_k = \pi_k(\eta)$. If $c_1(L_k) \geq -1$, it follows from Lemma 2.3.1 that D_{kk} is surjective for every k. Hence in this case, the above matrix equation has a solution ξ for every $\eta \in \Omega^{0,1}(S^2, E)$. If $c_1(L_k) < -1$ for some k then D_{kk} is not surjective and hence neither is D. □

Corollary 2.3.3. Let W be the product of S^2 with a symplectic manifold (M,ω) and $A \in H_2(W;\mathbb{Z})$ be the homology class represented by the sphere $S^2 \times \{\bullet\}$. Then, for every $J \in \mathcal{J}(M,\omega)$, the product almost complex structure $\widetilde{J} := i \times J$ defined via the natural direct sum decomposition $T_{(z,p)}W = T_z S^2 \oplus T_p M$, is regular for A.

Proof. Identify S^2 with the Riemann sphere $\mathbb{C} \cup \{\infty\}$ with its standard complex structure i. Every $\widetilde{J}$-holomorphic A-sphere has the form $u(z) = (\phi(z), x_0)$, where ϕ is a fractional linear transformation and $x_0 \in M$. Indeed, a map $u = (u_S, u_M) : S^2 \to S^2 \times M$ is $\tilde{J}$-holomorphic iff $u_S : S^2 \to S^2$ is holomorphic and $u_M : S^2 \to M$ is J-holomorphic. If $[u] = A_0 = [S^2 \times \{\bullet\}]$, then we also have

$$[u_S] = [S^2], \qquad \text{and} \qquad [u_M] = 0.$$

The latter implies that u_M has zero energy as a J_M-holomorphic curve in M, i.e., $\int_{S^2} u_M^* \omega = \langle [\omega], [u_M] \rangle = 0$, and hence u_M is a constant. Moreover, $u_S : S^2 \to S^2$ is a holomorphic map of degree 1, and thus is bi-holomorphic, so after a reparametrization of the domain we can assume $u_S = \mathrm{id}$. Thus, up to reparametrization, any $\tilde{J}$-holomorphic curve of the above form corresponds to a map $z \mapsto (z,m)$ for $m \in M$. There is a natural splitting of complex vector bundles

$$u^* TW = TS^2 \oplus E_0^{(n-1)}$$

where $E_0^{(n-1)}$ denotes the trivial complex bundle of rank $n-1$ whose fiber at every point $z \in S^2$ is $(T_m M, J)$. Now u is regular iff the following operator is surjective

$$D_u : \mathcal{W}^{1,p}(u^* TW) \to \mathcal{L}^p(\Lambda^{0,1} \otimes_{\widetilde{J}} u^* TW).$$

We can make use of the natural splitting above to split the domain and target of D_u:

$$\mathcal{W}^{1,p}(u^* TW) = \mathcal{W}^{1,p}(TS^2) \oplus \mathcal{W}^{1,p}(E_0^{(n-1)}), \quad \text{and}$$

$$\mathcal{L}^p(\Lambda^{0,1} \otimes_{\widetilde{J}} u^* TW) = \mathcal{L}^p(\Lambda^{0,1} \otimes_i TS^2) \oplus \mathcal{L}^p(\Lambda^{0,1} \otimes_J E_0^{(n-1)}).$$

In light of the split nature of the nonlinear CR equation for $\tilde{J}$-holomorphic maps $u : S^2 \to S^2 \times M$, it turns out that the matrix form of D_u with respect to these splittings is

$$D_u = \begin{pmatrix} D_{S^2} & 0 \\ 0 & D_m \end{pmatrix}$$

where D_{S^2} is the natural Cauchy-Riemann operator defined by the holomorphic vector bundle structure of (TS^2, i), and D_m is the linearization of $\overline{\partial}_J$ at the constant J-holomorphic sphere $S^2 \to M, z \mapsto m$. For the case of a constant map, we see that the latter is just the standard Cauchy-Riemann operator on the trivial bundle $E_0^{(n-1)}$ i.e., it is the operator determined by the unique holomorphic structure on $E_0^{(n-1)}$ for which the constant maps are holomorphic. As such this operator splits further with respect to the splitting of $E_0^{(n-1)}$ into holomorphic line bundles determined by any complex basis of $T_m M$. This yields a presentation of D_u in the form

$$D_u = \begin{pmatrix} D_{S^2} & 0 & \dots & 0 \\ 0 & \overline{\partial} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \overline{\partial} \end{pmatrix},$$

where each of the diagonal terms are complex-linear Cauchy-Riemann type operators on line bundles, with the $\overline{\partial}$ entries in particular denoting operators that are equivalent to the standard operator

$$\overline{\partial} : \mathcal{W}^{1,p}(S^2, \mathbb{C}) \to \mathcal{L}^p(\Lambda^{0,1} \otimes_i \mathbb{C}), \qquad f \mapsto df + i\, df \circ i.$$

These operators are surjective by Lemma 2.3.2, since $c_1(E_0^1) = 0 > -1$. Similarly, D_{S^2} is also surjective since $c_1(TS^2) = 2 > -1$. $\square$

Chapter 3

Compactness

For various geometric purposes, it would be nice if our moduli spaces of holomorphic curves were compact. However, this is not always true. For example, when $\Sigma = S^2$, the manifold $\mathcal{M}(A; J)$ itself cannot be compact (unless it consists of constant maps) since the non-compact group $G = PSL(2, \mathbb{C})$ of biholomorphic automorphisms of S^2 acts on this space by reparametrization of the domain. However, in some cases the moduli space $\mathcal{M}(A; J)/G$ of unparametrized J-holomorphic spheres is compact.

If J is tamed by a symplectic form ω then it follows from the energy identity (Lemma 1.4.1) that there is uniform bound on the energy, and hence on the $\mathcal{W}^{1,2}$-norm, of all J-holomorphic curves in a given homology class. This is the Sobolev borderline case, and as a result, the space of such curves will in general not be compact. In fact, the standard elliptic bootstrapping argument asserts that any sequence u^ν in $\mathcal{M}(A; J)$ which is bounded in the $\mathcal{W}^{1,p}$-norm for some $p > 2$ has a subsequence which converges uniformly with all derivatives. On the other hand, if the first derivatives of u^ν are only bounded in $\mathcal{L}^2$ then the conformal invariance of the energy in two dimensions leads to the phenomenon of bubbling, which was first discovered by Sacks and Uhlenbeck in the context of minimal surfaces. Indeed, a simple geometric argument using conformal rescaling allows one to construct a J-holomorphic map $v : \mathbb{C} \to M$ with finite area which, by *removal of singularities*, can be extended to $S^2 = \mathbb{C} \cup \{\infty\}$. This is the phenomenon of *bubbling off of spheres*, which we shall study in the following sections.

3.1 Sequences with uniformly bounded energy

The energy identity distinguishes holomorphic curves in symplectic manifolds from those in general almost complex manifolds. In the former case, the energy is a topological invariant, while in the latter case, the energy of curves in a certain homology class does not satisfy a universal bound and hence no general compactness results are available.

We saw in Lemma 1.4.1 that the energy of a holomorphic curve $u : (\Sigma, j_\Sigma, \mathrm{d}\,\mathrm{vol}_\Sigma) \to (M, \omega, J)$ (where $J \in \mathcal{J}_\tau(M, \omega)$ is given by

$$E(u) = \frac{1}{2}\int_\Sigma |\,\mathrm{d}u|_J^2 \,\mathrm{d}\,\mathrm{vol}_\Sigma = \int_\Sigma u^*\omega.$$

Thus, we obtain a uniform $\mathcal{L}^2$-bound on the first derivatives for all holomorphic curves representing the same homology class. Such $\mathcal{L}^2$ bounds do not suffice to establish compactness, however, as we shall see in Section 3.2. On the other hand, if we have a uniform $\mathcal{L}^p$-bound on the first derivatives of the elements of a sequence u_ν of holomorphic curves then standard “elliptic bootstrapping” techniques can be used to establish the existence of a convergent subsequence:

Theorem 3.1.1. Let (M, J) be a compact almost complex manifold and J^ν be a sequence of almost complex structures on M that converges in the $\mathscr{C}^\infty$-topology to J. Moreover, let $(\Sigma, j_\Sigma, \mathrm{d}\,\mathrm{vol}_\Sigma)$ be an open Riemann surface, $\Omega^\nu \subseteq \Sigma$ be an increasing sequence of open sets that exhaust Σ, and $u^\nu : \Omega^\nu \to M$ be a sequence of J^ν-holomorphic curves that

$$\sup_\nu \|du^\nu\|_{\mathcal{L}^\infty(K)} < \infty$$

for every compact subset $K \subseteq \Sigma$. Then u^ν has a subsequence which converges uniformly with all derivatives on compact subsets of Σ to a J-holomorphic curve $u : \Sigma \to M$.

The above theorem continues to hold when the maps u^ν are uniformly bounded in the $\mathcal{W}^{1,p}$-norm for some $p > 2$. However, the energy identity only guarantees a uniform bound $\mathcal{W}^{1,2}$-bound. Although this does not imply that the derivatives are uniformly bounded, it does give some control over what can happen when the derivatives blow up. As we shall see later, the conformal invariance of energy leads in this case to the formation of *bubbles*.

3.1.1 Removal of singularities

We now formulate the removable singularity theorem, which is an important analytical tool for understanding the bubbling phenomenon. In the integrable case, this is a familiar result from complex analysis. Its proof for J-holomorphic curves in symplectic manifolds will be carried out in Section 3.4. We give here a proof of continuity in the ω-compatible case using the monotonicity property of minimal surfaces.

We denote by $B \subseteq \mathbb{C}$ the closed unit disc and by $\mathbb{H} \subseteq \mathbb{C}$ the upper half plane.

Theorem 3.1.2 (Removal of singularities)**.** Let (M,ω) be a compact symplectic manifold, and J be an ω-tame almost complex structure on M with associated metric g_J. If $u : B \setminus \{0\} \to M$ is a J-holomorphic curve with finite energy $E(u) < \infty$ then u extends to a smooth map $B \to M$.

Proof (of continuity). Here we follow essentially the line of argument in Gromov's original work. We assume that J is compatible with ω so that the J-holomorphic curves minimize the energy. One can prove the removable singularity theorem in this case by using the monotonicity theorem for minimal surfaces. This states that there are constants $c > 0$ and $\varepsilon_0 > 0$ (which depend on M and the metric g_J) such that for every minimal surface S in (M, g_J) which goes through the point x

$$\mathrm{area}_{g_J}(S \cap B(x,\varepsilon)) \geq c\varepsilon^2$$

for $0 < \varepsilon < \varepsilon_0$. To apply this, suppose that $u(z)$ has two limit points p and q as $z \to 0$ and choose $\partial < d(p,q)/3$. Then the monotonicity theorem implies that each connected component of $u^{-1}(B(p,\partial))$ which meets $u^{-1}(B(p,\partial/2))$ is taken by u to a surface in M which has area $\geq c\partial^2/4$. Therefore, becuase the image of u is minmal and has finite area (or energy) $E(u)$, there an only be a finite number of such components. Similar remarks apply to q. Hence there exists an $r_0 > 0$ such that, for any $r < r_0$, the image γ_τ of the circle $\{z \in \mathbb{C} \mid |z| = r\}$ under u meets both $B(p,\partial)$ and $B(q,\partial)$, and so must have length $\ell(\gamma_r) > d(p,q) - 2\partial > \partial$. But the conformality of u implies that $|du| = \sqrt{2}|\partial u/\partial\theta|/r$, and we find that

$$\begin{aligned} E(u) &= \int_0^1 \int_{S^1} \frac{|\partial u/\partial\theta|^2}{r^2} r \,\mathrm{d}r \wedge \mathrm{d}\theta \\ &\geq \int_0^1 \left(\int_{S^1} |\partial u/\partial\theta| \,\mathrm{d}\theta \right)^2 \frac{1}{2\pi r} \,\mathrm{d}r \\ &= \int_0^1 \frac{\ell(\gamma_r)^2}{2\pi r} \,\mathrm{d}r \\ &\geq \int_0^{r_0} \frac{\partial^2}{2\pi r} \,\mathrm{d}r, \end{aligned}$$

which is impossible because $E(u)$ is finite. The geometric idea here is that, because of conformality, if the loops γ_r are long they must also stretch out in the radial direction, and hence form a surface of infinite area. □

Later, we shall see how the above theorems together with the energy identity can be used to understand the bubbling phenomenon for pseudoholomorphic curves. We shall see that bubbling, for a suitably chosen subsequence, can only occur near finitely many points. This follows from the fact that the energy of a non-constant J-holomorphic sphere cannot be arbitrarily small.

3.1.2 Energy quantization

Proposition 3.1.3. Let (M,J) be a compact almost complex manifold. Suppose that M is equipped with any Riemannian metric. Then there exists a constant $\hbar > 0$ such that

$$E(u) \geq \hbar$$

for every non-constant J-holomorphic sphere $u : S^2 \to M$.

Proof. The 2-sphere is conformally diffeomorphic to the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Hence it suffices to prove the assertions for finite energy holomorphic maps $u : \mathbb{C} \to M$. The apriori estimate of Lemma 3.3.1 below asserts that there exists a constant $\partial > 0$ such that

$$\int_{B_r(z)} |\mathrm{d}u|^2 < \partial \implies |\mathrm{d}u(z)|^2 \leq \frac{8}{\pi r^2} \int_{B_r(z)} |\mathrm{d}u|^2$$

for every J-holomorphic curve $u : \mathbb{C} \to M$ and every $z \in C$. If $E(u) < \partial$ then these estimates hold for every $r > 0$ and it follows that u is a constant. □

In the following we shall denote by $\hbar = \hbar(M, \omega, J)$ the largest constant which satisfies the requirements of the earlier proposition.

$$\hbar(M, \omega, J) := \inf_{\substack{u:S^2 \to M, \\ \bar{\partial}(u)=0,\ E(u)>0}} E(u).$$

3.2 The bubbling phenomenon

Let (M, ω) be a compact symplectic manifold and $J \in \mathcal{J}_\tau(M, \omega)$ be an ω-tame almost complex structure on M. Let $(\Sigma, j_\Sigma, \mathrm{d}\,\mathrm{vol}_\Sigma)$ be a closed Riemann surface. We saw in an earlier section that every sequence of J-holomorphic curves $u^\nu : \Sigma \to M$ with uniformly bounded first derivatives has a $\mathscr{C}^\infty$-convergent subsequence on compact sets. Hence compactness can only fail if the sequence $\mathrm{d}u_\nu$ is unbounded:

$$\sup_\nu \|\mathrm{d}u^\nu\|_{\mathcal{L}^\infty} = \infty.$$

If, in addition, the energy is uniformly bounded, i.e.,

$$\sup_\nu E(u^\nu) < \infty,$$

then a conformal rescaling argument shows that a holomorphic sphere *bubbles off*. Here's how this works. Let $z^\nu \in \Sigma$ be a point at which the real valued function $|\mathrm{d}u^\nu|$ attains its maximum:

$$|\mathrm{d}u^\nu(z^\nu)| = \|\mathrm{d}u^\nu\|_{\mathcal{L}^\infty(\Sigma)} =: c^\nu.$$

Passing to a subsequence, if necessary, we may assume that z^ν converges to a point $z_0 \in \Sigma$ and that c^ν diverges to infinity. Now choose a holomorphic coordinate chart $\phi : \Omega \to \Sigma$ defined on an open neighborhood $\Omega \subseteq \mathbb{C}$ of zero such that $\phi(0) = z_0$. Then the pullback volume form is

$$\phi^* \mathrm{d}\,\mathrm{vol}_\Sigma = \lambda^2\, \mathrm{d}s \wedge \mathrm{d}t$$

for some function $\lambda : \Omega \to (0, \infty)$. We may assume without loss of generality that $z^\nu \in \Omega$ for all ν and

$$\lambda(0) = 1, \quad 1/2 \leq \lambda(z) \leq 2$$

for all $z \in \Omega$. Consider the sequences

$$u^\nu_{\mathrm{loc}} := u^\nu \circ \phi : \Omega \to M, \quad z^\nu_{\mathrm{loc}} := \phi^{-1}(z^\nu).$$

In following we shall drop the subscript loc. Then

$$c^\nu = \frac{|\mathrm{d}u^\nu(z^\nu)|}{\lambda(z^\nu)} = \sup_\Omega \frac{|\mathrm{d}u^\nu|}{\lambda}, \quad \lim_{z \to \infty} z^\nu = 0.$$

Now choose $\varepsilon > 0$ such that $B_\varepsilon(z^\nu \subseteq \Omega$ for every ν and consider the reparametrized sequence $v^\nu : B_{\varepsilon c^\nu} \to M$ defined by

$$v^\nu(z) := u^\nu(z^\nu + z/c^\nu).$$

This sequence satisfies

$$|\mathrm{d}v^\nu(0) \geq 1/2, \quad \|\mathrm{d}v^\nu\|_{\mathcal{L}^\infty(B_{\varepsilon c^\nu})} \leq 2$$

and

$$E(v^\nu; B_{\varepsilon c^\nu}) = E(u^\nu; B_\varepsilon(z^\nu)) \le E(u^\nu),$$

where $E(u;B)$ denotes the energy of the restriction of u to B. The last identity follows from the conformal invariance of energy. Hence by the earlier theorem, there exists a subsequence, still denoted by v^ν, that converges uniformly with all derivatives on compact sets. The limit function $v : \mathbb{C} \to M$ is again a J-holomorphic curve such that

$$|\,\mathrm{d}v(0)| \ge 1/2, \qquad 0 < E(v) = \int_{\mathbb{C}} v^*\omega \le \sup_\nu E(u^\nu).$$

The conformal invariance of the energy implies that the map

$$\mathbb{C} \setminus \{0\} \to M : z \mapsto v(1/z)$$

has finite energy and so, by the removable singularity theorem, it extends smoothly over 0. Hence v extends to a non-constant J-holomorphic map from the Riemann sphere $S^2 \simeq \mathbb{C} \cup \{\infty\}$ to M. A J-holomorphic sphere v constructed in this way is called a *bubble*. The energy $E(v)$ of this bubble is positive and the energy of u^ν in an arbitrarily small neighborhood of the point z_0 is at leaast $E(v)$ in the limit $\nu \to \infty$, i.e.,

$$\liminf_{\nu\to\infty} E(u^\nu; B_\varepsilon(z_0)) \ge E(v) \tag{3.1}$$

for every $\varepsilon > 0$. To see this note that

$$\begin{aligned} E(v; B_R) &= \lim_{\nu\to\infty} E(v^\nu; B_R) \\ &= \lim_{\nu\to\infty} E(u^\nu; B_{R/c^\nu}(z^\nu)) \\ &\le \lim_{\nu\to\infty} E(u^\nu; B_\varepsilon(z_0)) \end{aligned}$$

for every $R > 0$ and every $\varepsilon > 0$. The inequality follows by taking the limit $R \to \infty$.

The results of this section suffice for several remarkable applications of J-holomorphic curves in symplectic topology that are due to Gromov, and in particular the non-squeezing theorem. These applications give a glimpse of the power of the theory of J-holomorphic curves in symplectic topology.

3.3 The mean value inequality

In this section, we shall mention the mean value inequality for the first derivative of a J-holomorphic curve with small energy. We denote by $B_r(z)$ the closed ball of radius r at z.

Lemma 3.3.1. Let (M,J) be a compact almost complex manifold, equipped with any Riemannian metric. Then there exists a constant $\partial > 0$ such that if $r > 0$ and $u : B_r \to M$ is a J-holomorphic curve, then

$$\int_{B_r} |\,\mathrm{d}u|^2 < \partial \implies |\,\mathrm{d}u(0)|^2 \le \frac{8}{\pi r^2}\int_{B_r} |\,\mathrm{d}u|^2.$$

The proof relies on a result about the partial differential inequality $\Delta w \ge -aw^2$ for the energy density $w := |\,\mathrm{d}u|^2/2$, where $\Delta = \partial_s^2 + \partial_t^2$ is the standard Laplacian on $\mathbb{R}^2$. The details are worked out in [2], pp. 81.

3.4 The isoperimetric inequality

The most basic isoperimetric inequality concerns simple closed curves in the Euclidean plane and is often expressed by saying that among all such curves of a given length the circle encloses the greatest area. Another way of saying this is as follows. If γ is a simple closed curve in $\mathbb{R}^2$ denote by $L(\gamma)$ its length with respect to the usual metric and by $\mathcal{A}(\gamma)$ the enclosed area. Then

$$\mathcal{A}(\gamma) \le \frac{1}{4\pi} L(\gamma)^2,$$

with equality iff γ is a circle. We show in this section that this form of the isoperimetric inequality has a natural extension to symplectic manifolds in which $\mathcal{A}(\gamma)$ is interpreted as the symplectic area of a disc with boundary γ.

In general (i.e., if ω does not vanish on $\pi_2(M)$) γ bounds many discs with different symplectic areas, and so we must specify the disc we consider. This is possible only if γ is sufficiently short.

Let (M,ω) be a compact symplectic manifold and $J \in \mathcal{J}_\tau(M,\omega)$ be an ω-tame almost complex structure. We assume throughout that M is equipped with the Riemannian metric determined by ω and J and we identify $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Consider a smooth loop $\gamma : S^1 \to M$ whose length

$$\ell(\gamma) := \int_0^{2\pi} |\dot\gamma(\theta)|_J \,\mathrm{d}t$$

is smaller than the injectivity radius of M. Then its image is contained in some geodesic ball $U \subseteq M$, whose radius is at most half the injectivity radius. Since this ball is contractible, γ admtis a smooth local extension $u_\gamma : B \to U$ such that

$$u_\gamma(e^{i\theta}) = \gamma(\theta)$$

for every $\theta \in \mathbb{R}$. We define the **local symplectic action** of γ as minus the symplectic area of u_γ and denote it by

$$a(\gamma) := -\int_B u_\gamma^*\omega.$$

Since any two such extensions (with images in geodesic balls of radii at most half the injectivity radius) are homotopic relative to their boundary, they have the same symplectic area. Thus the local symplectic action $a(\gamma)$ is well defined provided that γ is sufficiently short. Note also that the restriction of ω to U is exact, so that we may also define the local symplectic action by

$$a(\gamma) := \int_{S^1} \gamma^*\lambda, \quad \lambda \in \Omega^1(U), \quad \mathrm{d}\lambda = -\omega.$$

That this agrees with the previous definition follows from the Stokes' theorem.

Theorem 3.4.1 (Isoperimetric inequality)**.** Let (M,w) be a compact symplectic manifold and $J \in \mathcal{H}_\tau(M,\omega)$ be an ω-tame almost complex structure on M. Then for every constant $c > 1/3\pi$ there exists a constant $\delta > 0$ such that

$$\ell(\gamma) < \delta \implies |a(\gamma)| \geq c\ell(\gamma)^2 \tag{3.2}$$

for every smooth loop $\gamma : S^1 \to M$.

Remark 3.4.2. It is easy to see that the isoperimetric inequality holds with *some* constant c. Define the local extension $u_\gamma : B \to M$ by

$$u_\gamma(re^{i\theta}) := \exp_{\gamma(0)}(r\xi(\theta)),$$

where $\xi(\theta) \in T_{\gamma(o)}M$ is determined by the condition $\exp_{\gamma(0)}(\xi(\theta)) = \gamma(\theta)$. To prove Eq. (3.2), note that

$$|\partial_r u_\gamma| = |\xi(\theta)| = d(\gamma(0), \gamma(\theta)) \leq \ell(\gamma)$$

and

$$|\partial_\theta u_\gamma| \leq c_1|\dot\xi(\theta)| \leq c_2|\dot\gamma(\theta)|$$

for all r and θ and some constants c_1 and c_2 depending only on the metric. Hence

$$|a(u_\gamma)| = \left| \int_0^{2\pi} \int_0^1 \omega(\partial_r u_\gamma, \partial_\theta u_\gamma \right| \leq c_3\ell(\gamma)^2,$$

whenever γ is shorter than half the injectivity radius.

The idea of the proof is to show by direct calculation that it holds in symplectic vector spaces then to deduce i for general symplectic manifolds by using Darboux charts. This approach works because the charts do not distort the action and distort the lengths of short curves only a little. The details are worked out in [2], pp. 87.

3.5 Removal of singularities

The removable singularity theorem asserts that every J-holomorphic curve on the punctured disc with values in a compact symplectic manifold extends smoothly to the whole disc provided that it has finite energy. We now give a proof of this result independent of the earlier continuity proof and which instead relies on the mean value inequality. The first step is to extend the isoperimetric inequality to the case when the extension u_γ is defined over the punctured disc. This is related to the monotonicity property of minimal surfaces. We denote by $\ell(\gamma)$ the length of a path or loop $\gamma : I \to M$.

Lemma 3.5.1. Let (M,ω) be a compact symplectic manifold, and $J \in \mathcal{J}_\tau(M,\omega)$ be an ω-tame almost complex structure. Let $u : B \setminus \{0\} \to M$ be a J-holomorphic curve on the punctured unit disc such that $E(u) < \infty$. Then, for every $c > 1/4\pi$, there is a constant $r_0 > 0$ such that

$$0 < r < r_0 \implies \frac{1}{2}\int_{B_r} |du|^2 \leq c\ell(\gamma_r)^2,$$

where $\gamma_r : \mathbb{R}/2\pi\mathbb{Z} \to M$ denotes the loop $\gamma_r(\theta) := u(re^{i\theta})$.

Its proof and the proof of conformal removal of singularity using the above can be found in [2], pp. 92.

3.6 Spheres with minimal energy

We now present a compactness result which is the foundation of Gromov's non-squeezing argument. A homology class $A \in H_2(M)$ is called **spherical** if it is in the image of the Hurewicz homomorphism $\pi_2(M) \to H_2(M)$. The set of spherical classes is denote $H_2^S(M)$. Fix a spherical homology class $A \in H_2(M;\mathbb{Z})$ such that

$$\langle[\omega],A\rangle = \hbar,$$

where $\hbar = \hbar(M,\omega,J)$ is the constant in the energy quantization Proposition 3.1.3. Let $\mathcal{M}(A;J)$ denote the moduli space of J-holomorphic spheres $u : S^2 \to M$ representing the class A. By the removable singularity theorem, we may think of a holomorphic sphere as a finite energy holomorphic curve $u : \mathbb{C} \to M$. Then the function $\mathbb{C} \setminus \{0\} \to M : z \mapsto u(1/z)$ extends smoothly over zero and so u extends to a J-holomorphic map from the Riemann sphere $\mathbb{C} \cup \{\infty\}$ to M. The group $G = PSL(2,\mathbb{C})$ of Mobius transformations acts on $\mathcal{M}(A;J)$ by reparametrization. We say that a sequence of J-holomorphic curves $u^\nu : \mathbb{C} \to M$ converges in the $\mathscr{C}^\infty$ topology on S^2 if both $u^\nu(z)$ and $u^\nu(1/z)$ converge uniformly with all derivatives on compact subsets of $\mathbb{C}$.

Theorem 3.6.1. Let (M,ω) be a compact symplectic manifold and fix $J \in \mathcal{J}_\tau(M,\omega)$. If $A \in H_2(M;\mathbb{Z})$ is a spherical homology class such that $\langle[\omega],A\rangle = \hbar$ then the moduli space $\mathcal{M}(A;J)/G$ is compact, i.e., for every sequence $u^\nu : S^2 \to M$ of J-holomorphic spheres representing the class A there exists a sequence $\phi^\nu \in G$ such that the sequence $u^\nu \circ \phi^\nu$ has a $\mathscr{C}^\infty$-convergent subsequence.

Proof. By composing with a sequence of Mobius transformations in $SO(3)$ we may assume without loss of generality that

$$|du^\nu(0)|_{FS} = \sup_{z\in\mathbb{C}} |du^\nu(z)|_{FS}$$

for every ν. Since $|du^\nu(z)|_{FS} = |du^\nu(z)|(1+|z|^2)$, this implies

$$|du^\nu(0)| = \sup_{z\in\mathbb{C}} |du^\nu(z)|.$$

By conformal rescaling at zero we may further assume that

$$|du^\nu(0)| = 1.$$

By Theorem 3.1.1, u^ν has a subsequence, still denoted by u^ν, which converges uniformly with all derivatives on compact subsets of $\mathbb{C}$ to a non-constant J-holomorphic curve $u : \mathbb{C} \to M$. Now consider the sequence $z \mapsto u^\nu(1/z)$. This sequence converges uniformly with all derivatives on compact subsets of $\mathbb{C} \setminus \{0\}$ to $u(1/z)$. We claim that the sequence $u^{\nu}(1/z)$ also converges uniformly near zero. Otherwise bubbling would occur, and by Eq. (3.1), this would imply

$$\liminf_{\nu\to\infty} E(u^{\nu};\mathbb{C}\setminus B_R)\geq \hbar$$

for every $R>0$. It would then follow that $E(u;B_R)=0$, in contradiction to the fact that u is non-constant. Hence the sequence u^{ν} converges to u in the $\mathscr{C}^{\infty}$ topology on all of S^2. □

Note that in the above theorem, if the J is also varying, that is, if one has a sequence of J^{ν} holomorphic curves u^{ν}, exactly the same argument would show the existence of a convergent subsequence. This shows us that cobordisms $\mathcal{M}(A;\{J_\lambda\})$ are compact cobordisms when A is indecomposable.

Chapter 4

Gromov's non-squeezing theorem

Gromov's non-squeezing theorem is a cornerstone of symplectic topology. It says that a symplectomorphism cannot squeeze a ball into a cylinder of smaller radius.

Theorem 4.0.1 (Gromov, 1985)**.** If $\iota : (B^{2n}(r), \omega_{\text{std}}) \hookrightarrow (B^2(R) \times \mathbb{R}^{2n-2}, \omega_{\text{std}})$ is a symplectic embedding, then $r \leq R$.

Almost surprisingly, although the statement of the theorem has no mention of J-holomorphic curves, Gromov masterfully proved the above with the foundations of the parametric moduli space and it's compactness. Since the theory of J-holomorphic curves is generally easier to work with in closed manifolds, the first step is to transform this into a problem invoking embeddings into closed symplectic manifolds. Towards that end, consider the following extension of the non-squeezing theorem.

Theorem 4.0.2. Let (M, ω) be a closed symplectic manifold of dimension $2n-2 \geq 2$, which is aspherical (i.e., $\pi_2(V) = 0$). If there is a symplectic embedding of the ball $(B^{2n}(r), \omega_{\text{std}})$ into $B^2(R) \times M$ then $r \leq R$.

Proof (of Theorem 4.0.1). A symplectic embedding $B^{2n}(r) \hookrightarrow B^2(R) \times \mathbb{R}^{2n-2}$ induces a symplectic embedding $B^{2n}(r) \hookrightarrow B^2(R) \times M$, where $M := \mathbb{R}^{2n-2}/\lambda\mathbb{Z}^{2n-2}$ is the $(2n-2)$-torus and $\lambda > 0$ is sufficiently large. Hence, by Theorem 4.0.2, $R \geq r$. □

4.1 Proof using the monotonicity formula

This proof is essentially Gromov's original proof, and it depends on a compactness result (Theorem 3.6.1) that is one of the simplest applications of Gromov's compactness theorem, but can be proved without developing the compactness theory in its full generality. The following discussion follows [6], Section 5.1.

We shall assume $r \geq R$ and argue by contradiction. First, let's adapt the target to become a closed manifold. Choose a small $\varepsilon > 0$ and scale the usual area form on the sphere S^2, to get a form σ such that $\int_{S^2} \sigma = \pi(R+\varepsilon)^2$. Then there is a symplectic embedding $(B^2(R), \omega_{\text{std}}) \hookrightarrow (S^2, \sigma)$ and hence also $(B^2(R) \times \mathbb{R}^{2n-2}, \omega_{\text{std}}) \hookrightarrow (S^2 \times \mathbb{R}^{2n-2}, \sigma \oplus \omega_{\text{std}})$. Composing this with ι as above, we may regard ι as a symplectic embedding

$$\iota : (B^{2n}(r), \omega_{\text{std}}) \hookrightarrow (S^2 \times \mathbb{R}^{2n-2}, \sigma \oplus \omega_{\text{std}}).$$

Theorem 4.1.1. Suppose (M, ω) is a closed symplectic manifold of dimension $2n-2 \geq 2$, which is aspherical, σ is the area form on S^2, and there is a symplectic embedding $\iota : (B^{2n}(r), \omega_{\text{std}}) \hookrightarrow (S^2 \times M, \sigma \oplus \omega)$. Then

$$\pi r^2 \leq \int_{S^2} \sigma = \pi(R+\varepsilon)^2.$$

We'll prove this as a corollary of the following two results. The first has its origins in the theory of minimal surfaces and is a special case of much more general results. The second one will require us to apply the technical machinery developed earlier.

Theorem 4.1.2 (Monotonicity)**.** Suppose $r_0 > 0$, (Σ, j) is a Riemann surface and $u : (\Sigma, j) \to (B^{2n}(r), i)$ is a non-constant proper holomorphic map whose image contains 0. Then $\forall\, r \in (0, r_0)$,

$$\int_{u^{-1}(\overline{B^{2n}}(r))} u^*\omega_{\text{std}} \geq \pi r^2.$$

The crucial result in the proof is the following.

Proposition 4.1.3. There exists a compatible $J \in \mathcal{J}(S^2 \times M, \sigma \oplus \omega)$ with $\iota^* J = i$ on $B^{2n}(r)$ and a J-holomorphic sphere

$$u : S^2 \to S^2 \times M$$

with $[u] = [S^2 \times \{\bullet\}] \in H^2(S \times M)$ whose image contains $\iota(0)$.

In the following, we shall consider the moduli space of $\mathcal{M}_{0,1}^A(J)$ of unparametrized J-holomorphic spheres with a single marked point. With a similar argument, it can be shown that (see Theorem 4.1.6) these moduli spaces also arise as smooth manifolds (the only extra part is the regularity of the evaluation maps). The compactness results follow analogously, in particular, it is true that $\mathcal{M}_{0,1}^{S^2\times\{\bullet\}}(J)$ is compact for suitable J.

Next, we discuss the truly nontrivial part of the proof above: why does the J-holomorphic sphere in the earlier proposition exist? Let $W := S^2 \times M$, and $A := [S^2 \times \{\bullet\}]$. This turns out to be true not just for a specific J but also for *generic* ω-compatible almost complex structures on W, and there is nothing special about the point $\iota(0)$, as *every* point in W is in the image of some J-holomorphic sphere homologous to A. Moreover, this is also true for a generic subset of the special class of almost complex structures that match the integrable complex structure $\iota_* i$ on $\iota(B^{2n}(r))$. We will not be able to find these J-holomorphic curves explicitly, as we have no concrete knowledge about the symplectic embedding $\iota : B^{2n}(r) \hookrightarrow W$ and thus cannot even write down an explicit expression for J having the desired property in $\iota(B^{2n}(r))$. Instead, we argue from more abstract principles by starting from a simpler almost complex structure, for which the holomorphic curves are easy to classify, and then using a deformation argument to sow that the desired curves for our more general data must also exist. This argument can be outlined as follows:

1. Find a special $J_0 \in \mathcal{J}(W, \omega)$ for which the moduli space $M_{0,1}^{A_0}(J_0)$ of J_0-holomorphic spheres homologous to $[S^2 \times \{\bullet\}]$ and with one marked point is easy to describe precisely: in particular, the curves in $\mathcal{M}_{0,1}^{A_0}(J_0)$ are all *Fredholm regular*, and the moduli space is a closed $2n$-dimensionnal manifold diffeomorphic to W, with a diffeomorphism provided by the natural evaluation map
$$\mathrm{ev} : \mathcal{M}_{0,1}^{A_0}(J_0) \to W, \qquad [(S^2, j, z, u)] \mapsto u(z).$$
2. Choose $J_1 \in \mathcal{J}(W, \Omega)$ with the desired property $\iota^*(J_1) = i$ and show that for a generic choice, the moduli space $\mathcal{M}_{0,1}^{A_0}(J_1)$ is also a smooth $2n$-dimensional manifold.
3. Choose a homotopy $\{J_t\}$ from J_0 to J_1 and show that for a generic such choice, the resulting *parametric moduli space* $\mathcal{M}_{0,1}^{A_0}(\{J_t\})$ is a smooth $(2n+1)$-dimensional manifold, which is the cobordism joining the moduli spaces corresponding to J_0 and J_1.
4. Since $\mathrm{ev} : \mathcal{M}_{0,1}^{A_0} \to W$ is a diffeomorphism, its $\mathbb{Z}_2$-mapping degree is 1, and the fact that ev extends naturally over the cobordism $\mathcal{M}_{0,1}^{A_0}(\{J_t\})$ implies that its restriction to the other boundary component also has $\mathbb{Z}_2$-degree 1. It follows that $\mathrm{ev} : \mathcal{M}_{0,1}^{A_0} \to W$ is surjective, so for every $p \in W$, there is a J_1-holomorphic sphere $u : S^2 \to W$ with $[u] = A_0$ and a point $z \in S^2$ such that $u(z) = p$.

We carry out the details now. The only part that cannot be proved using the tools developed already is the compactness of $\mathcal{M}_{0,1}^{A_0}(\{J_t\})$, which is incidentally the only place where the assumption $\pi_2(M) = 0$ is used.

4.1.1 Step 1: The moduli space for J_0.

Identify S^2 with the Riemann sphere $\mathbb{C} \cup \{\infty\}$ with its standard complex structure i, choose any $J_M \in \mathcal{J}(M, \omega)$, and define $J_0 \in \mathcal{J}(W, \Omega)$ via the natural direct sum decomposition $T_{(z,p)}W = T_z S^2 \oplus T_p M$, that is

$$J_0 := i \oplus J_M.$$

From Corollary 2.3.3, it follows that the moduli space $\mathcal{M}_{0,1}^{A_0}(J_0)$ can be identified with the following set:

$$\mathcal{M}_{0,1}^{A_0}(J_0) = \{(u_m, \zeta) \mid m \in M, \zeta \in S^2\}.$$

where we define the J_0-holomorphic maps

$$u_m : S^2 \to S^2 \times M : z \mapsto (z, m).$$

The evaluation map $\mathrm{ev} : \mathcal{M}^{A_0}_{0,1}(J_0) \to S^2 \times M$ then takes the form

$$\mathrm{ev}(u_m, \zeta) = (\zeta, m),$$

and is thus clearly a diffeomorphism.

We now imply that $\mathcal{M}^{A_0}_{0,1}(J_0)$ is a smooth manifold of dimension $2n$, and indeed, this is precisely the prediction made by the index formula (see Theorem 4.1.6), which gives

$$\text{expected}\dim \mathcal{M}^{A_0}_{0,1}(J_0) = 2(n-3) + 2c_1(A_0) + 2 = 2n$$

after plugging in the computation

$$c_1(A_0) = c_1(u_m^* T(S^2 \times M)) = c_1(TS^2) + c_1(E_0^{n-1}) = 2.$$

The above does *not* immediately imply that every curve in $\mathcal{M}^{A_0}_{0,1}(J_0)$ is Fredholm; in general only the converse of this statement is true. This is something one needs to be careful about since the choice of J_0 is definitively *non-generic*. This means that we cannot expect transversality to be achieved for general reasons, but must instead by checked explicitly. This turns out to be not so hard, simply because the curves $u_m(z) = (z, m)$ are so explicit. The details are worked out in Lemma 5.1.5, [6].

Remark 4.1.4. The above is an example of a general phenomenon often called "automatic transversality": it refers to various situations in which despite (or in this case even *because of*) a non-generic choice of J, transversality can be achieved by reducing it to a problem involving Cauchy-Riemann operators on line bundles and applying Lemma 2.3.1. The case above is unusually fortunate, as it is not often possible to split a given Cauchy-Riemann operator over a sum of line bundles in just the right way.

4.1.2 Step 2: Transversality for J_1.

From now on, assume the symplectic embedding $\iota : (B^{2n}(r), \omega_{\mathrm{std}} \to (W, \Omega)$ can be extended symplectically to a neighborhood of the closure $\overline{B^{2n}}(r)$; this can always be achieved by shrinking r slightly without violating the assumption $r > R$. Now consider the closed subspace of $\mathcal{J}(W, \Omega)$ defined by

$$\mathcal{J}(W, \Omega; \iota) := \{J \in \mathcal{J}(W, \Omega) \mid \iota^* J = i \text{ on } \overline{B^{2n}}(r)\},$$

in other words this is the space of all Ω-compatible almost complex structures on W which match the particular integrable complex structure $\iota_* i$ on the closed set $\iota(\overline{B^{2n}}(r))$.

As with J_0 in the previous subsection, the condition $\iota^* J = i$ is non-generic in some sense, but it turns out not to matter for our purposes:

Proposition 4.1.5. There exists a Baire subset $\mathcal{J}_{\mathrm{reg}}(W, \Omega; \iota) \subseteq \mathcal{J}(W, \Omega; \iota)$ such that for any $J \in \mathcal{J}(W, \Omega; \iota)$, all J-holomorphic spheres homologous to A_0 are Fredholm regular, hence $\mathcal{M}^{A_0}_{0,1}(J)$ is a smooth manifold of dimension $2n$.

Proof. We begin with the following observations.

1. The expected dimension of $\mathcal{M}^{A}_{g,m}(J)$ depends in general on g, m and A, but not on J, thus as earlier, expected $\dim \mathcal{M}^{A_0}_{0,1} = 2n$ also applies to $\mathcal{M}^{A_0}_{0,1}(J)$ for any J.
2. Every pseudoholomorphic curve $u : S^2 \to W$ homologous to A_0 is *simple*, as $A_0 = [S^2 \times \bullet]$ is not a positive multiple of any other homology class in $H_2(S^2 \times M)$.
3. For any $J \in \mathcal{J}(W, \Omega; \iota)$, there is no closed non-constant J-holomorphic curve $u : \Sigma \to W$ whose image lies entirely in $\iota(\overline{B^{2n}_r})$. If such a curve did exist, then $\iota^{-1} \circ u$ would be a non-constant closed i-holomorphic curve in $\mathbb{R}^{2n}$ and would thus have positive energy

$$\int_\Sigma (\iota^{-1} \circ u)^* \omega_{\mathrm{std}} > 0$$

but this is impossible since ω_{std} vanishes on every cycle in $\mathbb{R}^{2n}$.

The result now follows from Theorem 4.1.6, which is a variation of the construction of our usual unconstrained moduli space. The crucial point is that the set of perturbations allowed by $\mathcal{J}(W,\Omega;\iota)$ is still large enough to prove that the universal moduli space for somewhere injective curves is smooth, because every such curve necessarily has an injective point outside of $\iota(\overline{B_r^{2n}})$. □

Theorem 4.1.6. Suppose (M,ω) is a closed symplectic manifold , $U \subseteq M$ is an open subset with compact closure, and $J^{\text{fix}} \in \mathcal{J}(M,\omega)$. Then there exists a Baire subset $\mathcal{J}_{\text{reg}}(M,\omega;U,J^{\text{fix}}) \subseteq \mathcal{J}(M,\omega;U,J^{\text{fix}})$ such that for every $J \in \mathcal{J}_{\text{reg}}$, the space $\mathcal{M}_U^*(J)$ of J-holomorphic curves with injective points mapped into U naturally admits the structure of a smooth finite-dimensional manifold, and the evaluation map on this space is smooth.

For details on this result, please see Remark 2.2.3 and for a full proof, [6], Theorem 4.1.8. In light of this result, we can choose $J_1 \in \mathcal{J}_{\text{reg}}(W,\Omega;\iota)$ so that $\mathcal{M}_{0,1}^{A_0}(J_1)$ is a smooth manifold of dimension $2n$.

4.1.3 Step 3: Homotopy of almost complex structures.

Denote by $\mathcal{J}(W,\Omega;J_0,J_1)$ the space of smooth Ω-compatible homotopies between J_0 and J_1, i.e., this consists of all smooth 1-parameter families $\{J_t\}_{t\in[0,1]}$ such that $J_t \in \mathcal{H}(W,\Omega)$ for all $t \in [0,1]$ and J_t matches the structures chosen above for $t = 0,1$. This gives rise to the parametric moduli space

$$\mathcal{M}_{0,1}^{A_0}(\{J_t\}) = \{(u,t) \mid t \in [0,1], u \in \mathcal{M}(J_t)\}.$$

The following is the fundamental input we need from the compactness theory of holomorphic curves. It depends on certain topological detalis in the setup we've chosen, and in particular on the fact that $A_0 = [S^2 \times \{\bullet\}]$ is a primitive homology class and $\pi_2(M) = 0$.

Proposition 4.1.7. For any $\{J_t\} \in \mathcal{J}(W,\Omega;J_0,J_1)$, $\mathcal{M}_{0,1}^{A_0}(J_1)$ is compact.

The above follows from the discussion in Section 3.6.

Notice that since $\mathcal{M}_{0,1}^{A_0}(J_1)$ is naturally a closed subset of $\mathcal{M}_{0,1}^{A_0}(\{J_t\})$ and is already known to be a smooth manifold, this implies that $\mathcal{M}_{0,1}^{A_0}$ is a closed manifold. Since Fredholm regularity is an open condition, the same is then true for all $\mathcal{M}_{0,1}^{A_0}$ with t in some neighborhood of either 0 or 1, and for t in this range the natural projection

$$\mathcal{M}_{0,1}^{A_0}(\{J_t\}) \to \mathbb{R} : (u,t) \mapsto t$$

is a submersion. We cannot expect this to be true for all $t \in [0,1]$, not even for a generic choice of the homotopy, but we can at least arrange for $\mathcal{M}_{0,1}^{A_0}(\{J_t\})$ to carry a smooth structure. The following is the parametric version of Theorem 4.1.6 for unparametrized curves with a marked point.

Proposition 4.1.8. There exists a Baire subset

$$\mathcal{J}_{\text{reg}}(W,\Omega;J_0,J_1) \subseteq \mathcal{J}(W,\Omega;J_0,J_1)$$

such that for any $\{J_t\} \in \mathcal{J}_{\text{reg}}(W,\Omega;J_0,J)1)$, $\mathcal{M}_{0,1}^{A_0}$ is a compact smooth manifold, with boundary

$$\partial\mathcal{M}_{0,1}^{A_0}(\{J_t\}) = \mathcal{M}_{0,1}^{A_0}(J_0) \cup \mathcal{M}_{0,1}^{A_0}(J_1).$$

4.1.4 Step 4: Conclusion of the proof.

We will now derive the desired existence result using the $\mathbb{Z}_2$-mapping degree of the evaluation map. In general, if X and Y are closed and connected orientable n-dimensional manifolds and $f : X \to Y$ is a continuous map, then the degree $\deg_2(f) \in \mathbb{Z}_2$ can be defined by the condition

$$f_*[X] = \deg_2(f)[Y] \in H_n(Y;\mathbb{Z}_2),$$

where $[X] \in H_n(X;\mathbb{Z}_2)$ and $[Y] \in H_n(Y;\mathbb{Z}_2)$ denote the respective fundamental classes with $\mathbb{Z}_2$-coefficients. Equivalently, if f is smooth then $\deg_2(f)$ can be defined as the modulo 2 count of points in $f^{-1}(t)$ for a regular point y.

Choosing a generic homotopy $\{J_t\} \in \mathcal{J}_{\mathrm{reg}}(W, \Omega; J_0, J_1$, the parametric moduli space $\mathcal{M}_{0,1}^{A_0}(\{J_t\})$ now furnishes a smooth cobordism between the two closed manifolds $\mathcal{M}_{0,1}^{A_0}(J_0)$ and $\mathcal{M}_{0,1}^{A_0}(J_1)$. It is a compact cobordism by the discussion of the next section. Consider the evaluation map

$$\mathrm{ev} : \mathcal{M}_{0,1}^{A_0}(\{J_t\}) \to W : ([(S^2, j, z, u)], t) \mapsto u(z),$$

and denote its restriction to the two boundary components by $\mathrm{ev}_0 : \mathcal{M}_{0,1}^{A_0}(J_0) \to W$ and $\mathrm{ev}_1 : \mathcal{M}_{0,1}^{A_0}(J_1) \to W$. As we say earlier, ev_0 is a diffeomorphism, thus $(\mathrm{ev}_0)_*[\mathcal{M}_{0,1}^{A_0}(J_0)] = [W] \in H_{2n}(W; \mathbb{Z}_2)$. It follows that

$$(\mathrm{ev}_1)_*[\mathcal{M}_{0,1}^{A_0}(J_1)] = [W] \in H_{2n}(W; \mathbb{Z}_2)$$

as well, hence $\deg_2(\mathrm{ev}_1) = 1$ and ev_1 is therefore surjective. In particular, $\mathrm{ev}_1^{-1}(\iota(0))$ is not empty, and thus our proof is complete.

4.2 Proof using symplectic blowup

We now give a modification of the previous section's argument using the symplectic blowup construction to circumvent the monotonicity theorem for minimal surfaces.

4.2.1 The blowup construction

The naive idea of blowing up a complex manifold (M, J) at a point x_0 is to replace the point x_0 by the set of all complex lines (tangent directions) through x_0. To make this definition more formula, consider the total space L of the tautological line bundle over $\mathbb{C}P^{n-1}$:

$$L := \{(z, [w_1 : \cdots : w_n]) \in \mathbb{C}^n \times \mathbb{C}P^{n-1} \mid z \in \mathbb{C}w\}.$$

The map $(z, [w_1 : \cdots : w_n]) \mapsto z$ identifies the complement of the zero section in L with $\mathbb{C}^n \setminus \{0\}$. Thus L is a copy of $\mathbb{C}^n$ in which the origin 0 has been replaced by the zero section of L. Since the zero section is the set of all lines in $\mathbb{C}^n$ through 0, we may think of L as a model for the blowup of $\mathbb{C}^n$ at 0.

To blow up a general complex manifold, one proceeds as follows. Choose a complex embedding $\iota : B^{2n}(\delta) \hookrightarrow M$ such that $\iota(0) = x_0$ and consider the r-ball sub-bundle U_r of L:

$$U_r := \{(z, [w_1 : \cdots : w_n] \in \mathbb{C}^n \times \mathbb{C}P^{n-1} \mid z \in \mathbb{C}w, |z| < r\}.$$

Now define the blowup $\widetilde{M}$ as the union

$$\widetilde{M} := (M \setminus \{x_0\}) \cup U_r / \sim,$$

where the equivalence relation identifies the point $(z, [z_1 : \cdots : z_n]) \in U_r \setminus \mathbb{C}P^{n-1}$ with the point $\iota(z) \in M$. Then $\widetilde{M}$ is a complex manifold and there is an embedding $\widetilde{\iota} : \mathbb{C}P^{n-1} \hookrightarrow \widetilde{M}$ defined as the composition of the embedding of the zero section $\mathbb{C}P^{n-1} \hookrightarrow U_r : [w_1 : \cdots : w_n] \mapsto (0, [w_1 : \cdots : w_n])$ with the inclusion $U_r \hookrightarrow \widetilde{M}$. The image of this embedding is the **exceptional divisor**

$$E := \widetilde{\iota}(\mathbb{C}P^{n-1}) \subseteq \widetilde{M}.$$

Moreover, there is a holomorphic projection $pi : \widetilde{M} \to M$ such that $E = \pi^{-1}(x_0)$ and π restricts to a diffeomorphism from $\widetilde{M} \setminus E$ to $M \setminus \{x_0\}$. We will call $(\widetilde{M}, J)$ the **complex blowup** of (M, J) at x_0. One can show that the biholomorphism class of the resulting manifold is independent of the choice of embedding ι.

We emphasize that every almost complex structure J on M such that $\iota^* J = J_0$ induces a unique almost complex structure $\widetilde{J}$ on $\widetilde{M}$ with respect to which the projection π is holomorphic. Another useful fact about complex blowup is that any J-holomorphic curve through x_0 lifts to the blowup. Since this is a local statement it suffices to consider the lift of non-constant J_0-holomorphic maps $u : (B^2, 0) \to (\mathbb{C}^n, 0)$. Every such map has the form

$$u(z) = z^k (h_1(z), \ldots, h_n(z)), \quad k \geq 1,$$

where $h_i(0) \neq 0$ for some i, and it lifts to

$$\tilde{u}(z) = (z^k h_1(z), \dots, z^k h_n(z), [h_1(z) : \cdots : h_n(z)]) \in L.$$

If (M, J) supports a symplectic form ω it is not hard to define a symplectic form $\tilde{\omega}$ on the blowup $\tilde{M}$ whose integral over the line in the exceptional divisor E is small. However, this approach obscures the close connection between symplectic blowing up and embedded balls in (M, ω). It was point out in McDuff [which paper] that the most geometric way to think of the symplectic blowup is as follows: cut out the interior of a symplectically embedded ball $B = \iota(B^{2n}(\rho)))$ and collapse the bounding sphere $\partial B \simeq S^{2n-1}$ via the Hopf map to an exceptional divisor $E \simeq \mathbb{C}P^{n-1}$ This gives rise to an alternative description of the blwuop as a (suitably smoothed) quotient

$$\widehat{M_\rho} := M \setminus \iota(\operatorname{int} B)/\sim,$$

where $\iota(z)\tilde{\iota}(\lambda z)$ for $z \in \partial B^{2n}(\rho)$ and $\lambda \in S^1$. In this formulation, the exceptional divisor is the quotient of $\iota(\partial B)$ by the S^1-action. Since this S^1 action preserves ω, the form ω descends to the quotient; in other words the exceptional divisor is the corresponding symplectic reduced space, which is none other than $\mathbb{C}P^{n-1}$ provided with a suitable multiple of the Fubini-Study form ω_{FS}. Thus the blowup $\widehat{M_\rho}$ inherits a symplectic form $\widehat{\omega_\rho}$ from M which agrees with ω on $M \setminus B$ and restricts on the exceptional divisor E to the form $\rho^2 \omega_{FS}$, where ω_{FS} is normalized so that it integrates to π on any line. We give precise formulas for $\widehat{\omega_\rho}$ below.

This description of the symplectic blowup shows its connection with embedded balls and hence makes plausible its relevance to the non-squeezing theorem. Note also that this process is reversible: given a symplectically embedded copy E of $(\mathbb{C}P^{n-1}, \rho^2 \omega_{FS})$ in (M, ω) whose normal line bundle can be identified with the canonical line one can cut out a small neighborhood of E and glue in a ball of appropriate radius $\rho + \varepsilon$ o obtain a symplectic manifold called the **blowdown** of (M, ω) along E. (Paradoxically, blowing up reduces the volume of M, while blowing down increases it. The terminology obviously comes from the blow up operation in the complex category which does enlarge the point x_0 to a whole submanifold.)

Proposition 4.2.1. Let (M, ω) be a symplectic manifold, $\iota : B^{2n}(r) \hookrightarrow M$ be a symplectic embedding, and $J \in \mathcal{J}(M, \omega)$ be an ω-compatible almost complex structure such that $\iota^* J = J_0$. Then, for every $\rho < r$, there exists a symplectic form $\widetilde{\omega_\rho}$ on the complex blowup $(\tilde{M}, \tilde{J})$ of (M, J) at $x_0 := \iota(0)$ with the following properties.

1. The 2-form $\pi^* \omega$ agrees with $\widetilde{\omega_\rho}$ on $\pi^{-1}(M \setminus \iota(B_r) \subseteq \tilde{M}$.
2. $\tilde{\iota}^* \widetilde{\omega_\rho} = \rho^2 \omega_{FS}$ is the Fubini-Study form (with integral $\rho^2 \pi$ over any line).
3. $\widetilde{\omega_\rho}$ is compatible with $\tilde{J}$.
4. For every smooth map $\tilde{u} : \Sigma \to \tilde{M}$,

$$\int_\Sigma (\pi \circ \tilde{u})^* \omega = \int_\Sigma \tilde{u}^* \widetilde{\omega_\rho} + \pi \rho^2 (\tilde{u} \cdot E).$$

Proof. We first show that in a neighborhood of the zero section in the tautological line bundle $L \to \mathbb{C}P^{n-1}$ is symplectomorphic to a spherical shell in $\mathbb{R}^{2n}$. We saw above that the δ-ball bundle U_δ has coordinates $(z_1, \dots, z_n, [z_1 : \cdots : z_n])$. Since the Fubini-Study form is, in homogeneous coordinates, given by $(i/2)\partial\bar{\partial}(\log |z^2|)$ t it follows that the 2-form

$$\begin{aligned}
\omega_\rho &= \frac{i}{2} \partial \bar{\partial} (|z|^2 + \rho^2 \log |z|^2) \\
&= \frac{i}{2} \left(dz \wedge d\bar{z} + \rho^2 \frac{dz \wedge d\bar{z}}{|z|^2} - \rho^2 \frac{\bar{z} \cdot dz \wedge z \cdot d\bar{z}}{|z|^4} \right)
\end{aligned}$$

extends to a form on U_δ that restricts to $\rho^2 \omega_{FS}$ on the exceptional divisor. Here we denote

$$dz \wedge d\bar{z} = \sum_j dz_j \wedge d\bar{z_j}, \qquad \bar{z} \cdot dz = \sum_j \bar{z_j} \cdot dz_j.$$

Now consider the diffeomorphism

$$h_\rho : B^{2n}(\delta) \setminus \{0\} \to B^{2n}(\sqrt{\rho^2 + \delta^2}) \setminus B^{2n}(\rho)$$

given by

$$h_\rho(z) := \sqrt{|z|^2+\rho^2}\,\frac{z}{|z|} = \sqrt{1+\frac{\rho^2}{|z^2|}}\,z.$$

A somewhat tedious but elementary calculation shows that the pullback of the standard symplectic form under h_ρ is given by

$$h_\rho^*\omega_0 = \frac{i}{2}\left(\mathrm{d}z\wedge\mathrm{d}\bar{z} + \rho^2\frac{\mathrm{d}z\wedge\mathrm{d}\bar{z}}{|z|^2} - \rho^2\frac{\bar{z}\cdot\mathrm{d}z\wedge z\cdot\mathrm{d}\bar{z}}{|z|^4}\right) = \omega_\rho.$$

This formula allows us to give a precise description of the blowup $(\widehat{M}_\rho, \widehat{\omega}_\rho)$. We assume that $\iota : B^{2n}(r)\to M$ is a symplectic embedding and, for any $\rho < r$, we define $\widehat{\omega}_\rho$ to equal ω on $M\setminus\iota(B^{2n}(\rho))$. This form extends smoothly over the exceptional divisor since we may identify the spherical shell $\iota(B^{2n}(r)\setminus B^{2n}(\rho))$ with $U_\varepsilon\setminus\mathbb{C}P^{n-1}$ via the map $h_\rho\circ\iota^{-1}$.

We now construct a diffeomorphism $f : \widetilde{M}\to\widehat{M}_\rho$ such that the pullback form

$$\widetilde{\omega}_\rho := f^*\widehat{\omega}_\rho$$

satisfies the requirements of the proposition. Choose $\delta > 0$ such that

$$\rho^2+\delta^2 < (r-\delta)^2$$

and let $\beta : [0,r]\to[\rho,r]$ be a smooth function such that

$$\beta(s) = \begin{cases}\sqrt{s^2+\rho^2}, & \text{for } 0\le s\le\delta,\\ s, & \text{for } r-\delta\le s\le r,\end{cases}$$

and

$$0 < s\le r \quad\Longrightarrow\quad 0<\beta'(s)\le 1.$$

Now define $f : \widetilde{M}\to\widehat{M}_\rho$ by

$$f(\widetilde{x}) := \begin{cases}\pi(\widetilde{x}) & \text{if } \pi(\widetilde{x})\in M\setminus\iota(B^{2n}(r-\delta)),\\ \iota\left(\frac{\beta(|z|)z}{|z|}\right), & \text{if } \pi(\widetilde{x}) = \iota(z),\ 0<|z|<r,\\ [\iota(w)], & \text{if } \widetilde{x} = \widetilde{\iota}([w]), [w]\in\mathbb{C}P^{n-1}.\end{cases}$$

By construction the restriction of the form $\widehat{\omega}_\rho$ to a deleted neighborhood of the exceptional divisor can be identified with the standard form ω_0 on the spherical shell $B^{2n}(\sqrt{\rho^2+\delta^2})\setminus B^{2n}(\rho)$. Therefore, the pullback form $\widetilde{\omega}_\rho := f^*\widehat{w}_\rho$ obviously satisfies assertions (i) and (ii). Moreover, it follows by direct calculation that the pullback of ω_0 under the map $z\mapsto\beta(|z|)z/|z|$ is compatible with J_0 whenever $\beta'(s)>0$ for $s>0$. This implies that $\widetilde{w}_\rho$ is compatible with $\widetilde{J}$.

To prove (4) assume first that $\widetilde{u}(\Sigma)\cdot E = 0$. Then a surgery construction along curves in E joining intersection points with opposite intersection numbers shows that we may assume $\widetilde{u}(\Sigma)\cap E=\emptyset$ without changing the homology class of $\widetilde{u}$ (though we increase the genus of Σ in the process of removing intersection points). It follows that $\widetilde{u}$ is homologous to a map with values in $M\setminus\iota(B^{2n}(r))\subseteq\widetilde{M}$. For any such map, assertion (4) is obvious. Thus we have proved (4) in the case $\widetilde{u}\cdot E = 0$. Now let $\widetilde{u} : \Sigma\to\widetilde{M}$ by any smooth map and denote $k := \widetilde{u}\cdot E$. Let $v : S^2\to\mathbb{C}P^{n-1}$ be a smooth map in the homology class $k[\mathbb{C}P^{n-1}]$ and denote $\widetilde{v} := \widetilde{\iota}\circ v : S^2\to\widetilde{M}$. Then

$$\widetilde{v}\cdot E = -k, \qquad \int_{S^2}\widetilde{v}^*\widetilde{\omega}_\rho = \rho^2\pi k, \qquad \int_{S^2}(\pi\circ\widetilde{v})^*\omega = 0.$$

Hence if $\widetilde{u}\#\widetilde{v}$ denotes the obvious map whose domain is the disjoint union of Σ with S^2 then $(\widetilde{u}\#\widetilde{v})\cdot E = 0$ and the main result follows. □

Proof (of Theorem 4.0.2). Fix a constant $\varepsilon > 0$, let $\sigma\in\Omega^2(S^2)$ be an area form $\int_{S^2}\sigma = \pi R^2+\varepsilon$, and choose an area preserving embedding of the ball $B^2(R)$ into S^2. Then the given symplectic embedding of $B^{2n}(r)$ into $B^2(R)\times M$ gives rise to a symplectic embedding

$$\iota : B^{2n}(r)\to W := S^2\times M.$$

Consider the blowup $(\widetilde{W}, \widetilde{\omega}_\rho)$ of the manifold W, determined as in Proposition 4.2.1 by the symplectic embedding ι and a constant $\rho < r$. Let $A := [S^2 \times \{\mathrm{pt}\}] \in H_2(W;\mathbb{Z})$, and denote its lift to $\widetilde{W}$ by $\widetilde{A} \in H_2(\widetilde{W};\mathbb{Z})$. Thus $\widetilde{A}$ is represented by the submanifold $\pi^{-1}(S^2 \times \{y\})$ for any $y \in M$ such that $x_0 := \iota(0) \notin S^2 \times \{y\}$.

Since A is J-indecomposable for every $J \in \mathcal{J}(W,\omega)$, Theorem 3.6.1 implies that the moduli space $\mathcal{M}(A;J)/G$ is compact for every $J \in \mathcal{J}(M,\omega)$. By choosing a product almost complex structure on $W = S^2 \times M$ we see that the evaluation map $ev : \mathcal{M}_{0,1}(A;J) \to W$ has degree one for some, and hence for every, $J \in \mathcal{J}_{\mathrm{reg}}$ (just like in the earlier argument). Since the moduli space is compact this implies that, for every $J \in \mathcal{J}(W,\omega)$ and every $x \in W$, there exists a J-holomorphic sphere in M which represents the class A and passes through x.

Now there exists an ω-compatible almost complex structure on W such that ι^*J is equal to the standard complex structure J_0 on $B^{2n}(r)$. Let $u_0 : S^2 \to M$ be a J-holomorphic sphere such that $[u_0] = A$ and $x_0 \in u_0(S^2)$. By discreteness of inverse image for J-holomorphic maps, the set $Z_0 = u_0^{-1}(x_0)$ is finite. consider the map

$$\widetilde{u_0} := \pi^{-1} \circ u_0 : S^2 \setminus Z_0 \to \widetilde{W}.$$

Because one can always lift holomorphic curves to a complex blowup, $\widetilde{u}_0$ extends to a a $\widetilde{J}$-holomorphic sphere in $\widetilde{W}$ which will still be denoted by $\widetilde{u}_0$. By construction, this extended J-holomorphic sphere satisfies $\widetilde{u}_0(Z_0) \subseteq E$. Hence by positivity of intersections,

$$\widetilde{u}_0 \cdot E > 0.$$

Now it follows from Proposition 4.2.1 (iv) that

$$\pi R^2 + \varepsilon = \mathrm{vol}(S^2) = \int_{S^2} u_0^* A\omega = \int_{S^2} (\pi \circ \widetilde{u}_0)^*\omega = \int_{S^2} \widetilde{u}_0^* \widetilde{\omega}_\rho + \pi\rho^2(\widetilde{u}_0 \cdot E) \geq \pi\rho^2.$$

Since this holds for every $\varepsilon > 0$ and every $\rho < r$ we deduce that $R \geq r$. □

4.3 Some generalizations and the symplectic camel

The first generalization to consider embedded balls of radius r in the product manifold

$$(W,\omega) = (S^2 \times M, \pi_1^*\sigma + \pi_2^*\tau),$$

where (M,τ) is an arbitrary symplectic manifold. If M is closed and we assume the existence of a J-holomorphic sphere through any point of an arbitrary closed symplectic manifold, then the previous argument goes through without essential change and yields the expected inequality. Even without this assumption, Lalonde-McDuff succeeded in proving this inequality by a rather complicated geometric construction. This construction applies to completely arbitrary, even non-compact, symplectic manifolds M.

A second generalization is to symplectic 2-sphere bundles

$$S^2 \hookrightarrow W \to M$$

over compact symplectic $(2n-2)$-manifolds. In this situation the same techniques apply and prove the non-squeezing inequality

$$\pi r^2 \leq \int_{S^2} \sigma$$

with A equal to the class of the fiber whenever there is a symplectic embedding of the standard ball $(B^{2n}(r), \omega_0)$ into (M,ω). This version of the non-squeezing theorem plays an important role in the work of Biran.

The notion of non-squeezing theorem is often, rather playfully, described as the *principle of symplectic camel*, since Ian Stewart referred to it by alluding to the parable of the *camel and eye of a needle*. The reference is to the fact that a camel (being larger) cannot pass through the eye of a needle, if it's motion was symplectically coherent. This has also attracted references in other surveys:

> Now, why do we refer to a symplectic camel in the title of this paper? This is because one can restate Gromov's theorem in the following way: there is no way to deform a phase space ball using canonical transformations in such a way that we can make it pass through a hole in a plane of conjugate coordinates x_j, p_j if the area of that hole is smaller than that of the cross-section of that ball.
> – Maurice A. de Gosson, *The symplectic camel and the uncertainty principle: the tip of an iceberg?*

Similarly,

> Intuitively, a volume in phase space cannot be stretched with respect to one particular symplectic plane more than its "symplectic width" allows. In other words, it is impossible to squeeze a symplectic camel into the eye of a needle, if the needle is small enough. This is a very powerful result, which is intimately tied to the Hamiltonian nature of the system, and is a completely different result than Liouville's theorem, which only interests the overall volume and does not pose any restriction on the shape.
> – Andrea Censi, *Symplectic camels and uncertainty analysis*

4.4 Epilogue: symplectic rigidity

The Gromov's non-squeezing theorem is related to the existence and properties of a certain symplectic invariant. Roughly speaking this invariant is the 2-dimensional size of a symplectic manifold just as the length spectra determined by geodesics measure the 1-dimensional size of a Riemannian manifold. They are very different from the volume. Gromov has suggested the terms *symplectic area* or *symplectic width*. However, the term *symplectic capacity*, even though intuitively closely related to volume, has now become widely used for this type of invariant.

The non-squeezing theorem is one of the basic geometric manifestations of *rigidity*. Indeed, Weinstein made the point that it can be considered as a geometric expression of the uncertainty principle. Given a point $(x_1, y_1, \dots, x_n, y_n)$ of $\mathbb{R}^{2n}$, think of x_i as the ith position coordinate and y_i as the ith momentum coordinate of some Hamiltonian system. By taking a measurement, we might find out that the state of the system lies somewhere in a subset U of $\mathbb{R}^{2n}$. Suppose that U is (or contains) a ball of radius r. Then the range of uncertainty in our knowledge of the values of a conjugate pair (x_i, y_i) is measured by the area πr^2, and the non-squeezing theorem can be thought of as saying that no matter how the system is transformed this range of uncertainty can never be made smaller.

Symplectic rigidity is also often stated in the form of the following result.

Theorem 4.4.1. For every symplectic manifold (M, ω), the group of symplectomorphisms of (M, ω) is $\mathscr{C}^0$-closed in the group of all diffeomorphisms.

Proofs have been given by Eliashberg-Gromov (which can be found in [3], Theorem 12.2.1) and Ekeland-Hofer. That symplectomorphisms are not dense in the group of volume preserving diffeomorphisms is atleast immediate from the statement of non-squeezing theorem.

The converse to the non-squeezing theorem, namely that the non-squeezing property characterizes symplectomorphism and anti-symplectomorphisms, was proved independently by Eliashberg and Ekeland-Hofer.

Appendix A

Fredholm theory

In this appendix, we talk about some functional analytic background needed in the proof of Theorem 2.1.2. The first section gives an introduction to compact operators and states some important results. The second section gives a brief overview of Fredholm operators while stating the foundational lemmas regarding Fredholm operators. Finally, the last two sections establish the proof of the implicit function theorem in the Banach space setting, along with the proof of the Sard-Smale theorem crucially used in the density of regular J in Theorem 2.1.2.

A.1 Compact operators

There is a special class of linear transformations between Banach spaces which generalize several properties of linear transformations between finite dimensional spaces.

Definition A.1.1. Let V and W be Banach spaces and let $T \in \mathcal{L}(V,W)$. If $\dim(\mathcal{R}(T))$ is finite, then we say that T is of finite rank. If the image of every bounded set in V is relatively compact in W, then we say that T is compact.

For compactness, it is sufficient to verify that $T(B)$ is relatively compact in W when B is the (closed) unit ball in V. Equivalently, $T \in \mathcal{L}(V,W)$ is compact if, and only if, given any bounded sequence $\{x_n\}$ in V, the sequence $\{T(x_n)\}$ admits a convergent subsequence in W. Further, since any bounded set in a finite dimensional space is relatively compact, it follows that every continuous linear transformation of finite rank is compact. Thus, in particular, linear transformations defined on finite dimensional spaces are compact.

Let X, Y and Z be Banach spaces and let $S \in \mathcal{L}(X,Y)$ and let $T \in \mathcal{L}(Y,Z)$. If one of these is compact, it is easy to see that their composition is also compact. In particular, since the identity map in an infinite dimensional Banach space cannot be compact, it follows that compact maps on infinite dimensional spaces are not invertible.

Theorem A.1.2 (Fredholm alternative)**.** Let V be a Banach space and let $T : V \to V$ be compact. Let $T^* : V^* \to V^*$ denote its adjoint. Then ($\mathcal{N}$, $\mathcal{R}$ denoting null space and range respectively),

1. $\mathcal{N}(I-T)$ is finite dimensional.
2. $\mathcal{R}(I-T)$ is closed and $\mathcal{R}(I-T) = \mathcal{N}(I-T^*)^{\perp}$.
3. $\mathcal{N}(I-T) = \{0\}$ iff $\mathcal{R}(I-T) = V$.
4. $\dim \mathcal{N}(I-T) = \dim \mathcal{N}(I-T^*)$.

The proof can be found in [7], 8.2.1.

The above theorem is the starting point of the theory of Fredholm operators, which we introduce in the next section.

A.2 Fredholm operators

A bounded linear operator $D : X \to Y$ between Banach spaces is called a **Fredholm operator** if it has finite dimensional kernel, a closed image[1], and a finite dimensional cokernel $Y/\operatorname{im}(D)$. The **index** of such an operator is defined as $\dim \ker D$ minus $\dim \operatorname{coker} D$. The following lemma is often used to show that a certain operator D is Fredholm.

Lemma A.2.1. Let X, Y, Z be Banach spaces. Assume that $D : X \to Y$ is a bounded linear operator and $K : X \to Z$ is a compact operator. Assume that there is a constant $c > 0$ such that

$$\|x\|_X \le c(\|Dx\|_Y + \|Kx\|_z)$$

[1]This is not actually required, that is, it is true that an operator with finite dimensional kernel and cokernel has closed image.

for $x \in X$. Then D has a closed image and finite dimensional kernel.

The proof is an easy exercise in functional analysis and can be found in [2], A.1.1. The above has a number of corollaries.

Corollary A.2.2. Let X and Y be Banach spaces and $D : X \to Y$ be a bounded linear operator with a closed image and finite dimensional kernel (resp. cokernel).

- For every compact operator $K : X \to Y$ the operator $D + K$ also has closed image and a finite dimensional kernel (resp. cokernel).
- There exists an $\varepsilon > 0$ such that if $P : X \to Y$ is a bounded linear operator with $\|P\| < \varepsilon$ then $D + P$ has a closed image and finite dimensional kernel (resp. cokernel).

The proof can be found in [2], pp. 494. The most important properties of Fredholm operators are related to their stability under perturbations. This follows from the following results:

Lemma A.2.3. Let X, Y and Z be Banach spaces.

1. A bounded linear operator $D : X \to Y$ is Fredholm iff there exists a bounded linear operator $T : Y \to X$ such that both $DT - \mathbb{1}$ and $TD - \mathbb{1}$ are compact operators.
2. If both $D : X \to Y$ and $T : Y \to Z$ are Fredholm operators prove that $TD : X \to Z$ is a Fredholm operator and $\operatorname{ind}(TD) = \operatorname{ind}(D) + \operatorname{ind}(T)$.
3. A bounded linear operator $D : X \to Y$ is Fredholm iff its dual operator $D^* : Y^* \to X^*$ is and that their indices are related by $\operatorname{ind}(D^*) = -\operatorname{ind}(D)$.

Theorem A.2.4. Let $D : X \to Y$ be a Fredholm operator.

- If $K : X \to Y$ is a compact operator then $D + K$ is a Fredholm operator and $\operatorname{ind}(D + K) = \operatorname{ind}(D)$.
- There exists an $\varepsilon > 0$ such that if $P : X \to Y$ is a bounded linear operator with $\|P\| < \varepsilon$ then $D + P$ is a Fredholm operator and $\operatorname{ind}(D + P) = \operatorname{ind}(D)$.

This means that the set of Fredholm operators is open with respect to the uniform operator topology and the index is constant on each component.

A.3 Implicit function theorem

Let X, Y be Banach spaces and $f : X \to Y$ be smooth map. For every $x \in X$, denote by $\mathrm{d}f(x) : X \to Y$ the differential of f at x. If this operator is bijective then its inverse $\mathrm{d}f(x)^{-1} : Y \to X$ is a bounded linear operator by the open mapping theorem. The inverse function theorem asserts that f has a local inverse near every point x at which $\mathrm{d}f(x)$ is invertible. More precisely,

Theorem A.3.1 (Inverse function theorem)**.** Let X and Y be Banach spaces, $U \subseteq X$ be an open set, and $f : U \to Y$ be continuously differentiable. Let $x_0 \in U$ and suppose that the differential $\mathrm{d}f(x_0) : X \to Y$ is bijective. Then there exists an open neighborhood $U_0 \subseteq U$ of x_0 such that the restriction of f to U_0 is injective, $V_0 := f(U_0)$ is an open subset of Y, $f^{-1} : V_0 \to U_0$ is continuously differentiable, and

$$\mathrm{d}f^{-1}(y) = \left(\mathrm{d}f(f^{-1}(y))\right)^{-1}$$

The proof is analogous to the proof in finite dimensions. See [8], section 17.7 and [2], pp. 501. It is based on the following crucial lemma about maps $\psi : X \to X$ whose derivative is close to the identity.

Lemma A.3.2. Let $\gamma < 1$ and R be positive real numbers. Let X be a Banach space, $x_0 \in X$, and $\psi : B_R(x_0) \to X$ be a continuously differentiable map such that

$$\|\mathbb{1} - \mathrm{d}\psi(x)\| \leq \gamma$$

for every $x \in B_R(x_0)$. Then ψ is injective and maps $B_R(x_0)$ onto an open set in X such that

$$B_{R(1-\gamma)}(\psi(x_0)) \subseteq \psi(B_R(x_0)) \subseteq B_{R(1+\gamma)}(x_0).$$

Moreover, the map $\psi^{-1} : \psi(B_R(x_0)) \to B_R(x_0)$ is continuously differentiable and

$$d\psi^{-1}(y) = d\psi(\psi^{-1}(y))^{-1}.$$

Hence, if ψ is of class $\mathscr{C}^\ell$ for some positive integer ℓ then so is ψ^{-1}.

The proof can be found in [2], Lemma A.3.2 and for later reference, we quote the following important equation from the proof.

$$(1-\gamma)\|x_1 - x_2\| \leq \|\psi(x_1) - \psi(x_2)\| \leq (1+\gamma)\|x_1 - x_2\| \tag{A.1}$$

for $x_1, x_2 \in B_R$.

A smooth map $f : X \to Y$ between Banach spaces is called **Fredholm** if the differential $df(x) : X \to Y$ is a Fredholm operator for every $x \in X$. Since the Fredholm index is invariant under small perturbations the index of $df(x)$ is independent of x. It will be denote by $\mathrm{ind}(f)$. For any smooth map $f : X \to Y$, Fredholm or not, a vector $y \in Y$ is called a **regular value** of f if $df(x) : X \to Y$ is onto and has a right inverse for every $x \in f^{-1}(y)$. The implicit function theorem asserts that $f^{-1}(y)$ is a smooth manifold for every regular value of f. Moreover, if f is a Fredholm map then the dimension of $f^{-1}(y)$ is finite and agrees with the Fredholm index of f.

Theorem A.3.3 (Implicit function theorem). Let X and Y be Banach spaces, $U \subseteq X$ be an open set, and ℓ be a positive integer. If $f : U \to Y$ is of class $\mathscr{C}^\ell$ and y is a regular value of f then $\mathcal{M} := f^{-1}(y) \subseteq X$ is a $\mathscr{C}^\ell$ Banach manifold and $T_x\mathcal{M} = \ker df(x)\ \forall\, x \in \mathcal{M}$. Hence, if f is a Fredholm map, $\mathcal{M}$ is finite dimensional and $\dim \mathcal{M} = \mathrm{ind}(f)$.

Before giving the proof, let us translate this geometric statement into more analytic language. Suppose without loss of generality that $y = 0$. If $x_0 \in M$ then, by assumption, the operator

$$D := df(x_0) : X \to Y$$

is surjective and has a right inverse $Q : Y \to X$ such that $DQ = \mathrm{id}_Y$. The existence of a right inverse is equivalent to the existence of a splitting

$$X = \ker D \oplus \mathrm{im}\, Q.$$

Here $\ker D$ consists of the solutions of the linearized equation $df(x_0)\xi = 0$, and we expect the space of solutions of the full nonlinear equation $f(x) = 0$ to *look like* the kernel of D locally near x_0. The implicit function theorem makes this precise. Its claim that $\ker D$ is tangent to $\mathcal{M} = f^{-1}(0)$ is equivalent to saying that there is a smooth map

$$\phi : \ker D \to Y$$

with $d\phi(0) = 0$ such that, if x is sufficiently close to x_0, then $f(x) = 0$ iff x has the form

$$x = x_0 + \xi + Q\phi(\xi), \qquad D\xi = 0,$$

for some sufficiently small vector ξ. Thus, if x_1 denotes $x_0 + \xi$, we are looking for a solution for $f(x_1 + \eta) = 0$ for some $\eta \in \mathrm{im}\, Q$. All we know is that there is an approximate solution of the equation $f(x) = 0$ (at the point $x = x_1$); we do not assume that $f(x_0) = 0$. Nevertheless, the next proposition says that if $f(x_1)$ is sufficiently small there is a true solution on the coset $C = x_1 + \mathrm{im}\, Q$.

Proposition A.3.4 (Quantitative version). Let X, Y be Banach spaces, $U \subseteq X$ be an open set, and $f : Y \to Y$ be a continuously differentiable map. Let $x_0 \in U$ be such that $D := df(x_0) : X toY$ is surjective and has a (bounded linear) right inverse $Q : Y \to X$. Choose positive constants δ and c such that $\|Q\| \leq c$, $B_\delta(x_0; X) \subseteq U$, and

$$\|x - x_0\| < \delta \qquad \Longrightarrow \qquad \|df(x) - D\| \leq 1/2c.$$

Suppose that $x_1 \in X$ satisfies

$$\|f(x_1)\| < \delta/4c, \qquad \|x_1 - x_0\| < \delta/8. \tag{A.2}$$

Then there exists a unique $x \in X$ such that

$$f(x) = 0, \qquad x - x_1 \in \mathrm{im}\, Q, \qquad \|x - x_0\| < \delta. \tag{A.3}$$

Moreover, $\|x - x_1\| \leq 2c\|f(x_1)\|$.

Proof. Define the map $h : U \to Y$ by
$$h(x) := f(x) - D(x - x_1).$$
Then
$$x + Qh(x) = x_1 \qquad \Longleftrightarrow \qquad f(x) = 0, \;\; x - x_1 \in \operatorname{im} Q.$$
To see this, suppose first that $x + Qf(x) - QD(x - x_1) = x_1$. Then $x - x_1 \in \operatorname{im} Q$. Hence $QD(x - x_1) = x - x_1$ and so $Qf(x) = 0$. Since Q is injective, this means $f(x) = 0$. The inverse is similar.

Therefore the problem reduces to finding a solution $x \in B_\delta(x_0; X)$ of the equation $x + Qh(x) = x_1$. Let us rewrite this equation in the form $\psi(x) = x_1$, where $\psi : U \to X$ is defined by
$$\psi(x) = x + Qh(x) = x + Q(f(x) - D(x - x_1)).$$
We shall see that ψ satisfies the hypotheses of the lemma, and that the point x_1 belongs to the image of $B_\delta(x_0; X)$ under ψ.

More precisely, $\mathrm{d}\psi(x) - \mathbb{1} = Q(\mathrm{d}f(x) - D)$, and hence it follows from the first equation that
$$\|x - x_0\| < \delta \qquad \Longrightarrow \qquad \|\mathbb{1} - \mathrm{d}\psi(x)\| \le c\|\mathrm{d}f(x) - D\| \le 1/2.$$
By Lemma A.3.2, ψ maps $B_\delta(x_0; X)$ bijectively onto some open set in X and
$$B_{\delta/2}(\psi(x_0); X) \subseteq \psi(B_\delta(x_0; X)) \subseteq \psi(B_{2\delta}(x_0; X)).$$
Now observe that by Eq. (A.1) and Eq. (A.3),
$$\begin{aligned} \|x_1 - \psi(x_0)\| &= \|\psi(x_1) - \psi(x_0) - Qf(x_1)\| \\ &= 2\|x_1 - x_0\| + c\|f(x_1)\| \\ &< \delta/2. \end{aligned}$$
Hence there is a unique element $x \in x_1 + \operatorname{im} Q$ such that $f(x) = 0$. Moreover, the inequality in the proof of lemma shows that
$$\|x - x_1\| \le 2\|\psi(x) - \psi(x_1)\| = 2\|Qf(x_1)\| \le 2c\|f(x_1)\|.$$
□

Proof (of original theorem). Assume without loss of generality that $y = 0$. Let $x_0 \in U$ such that $f(x_0) = 0$ and denote $D := \mathrm{d}f(x_0)$. By assumption this operator has a right inverse $Q : Y \to X$. Choose constants c and δ as in the earlier proposition. Shrinking δ, if necessary, we may also assume that
$$\|xi\| < \delta/8 \qquad \Longrightarrow \qquad \|f(x_0 + \xi) - f(x_0) - \mathrm{d}f(x_0)\xi\| < 2\|xi\|/c.$$
Let $\xi \in \ker D$ with $\|\xi\| < \delta/8$. Then $x_1 := x_0 + \xi$ satisfies Eq. (A.3). Hence, there is a unique element $x = x(\xi) \in X$ such that
$$f(x) = 0, \qquad x - x_0 - \xi \in \operatorname{im} Q, \qquad \|x - x_0\| < \delta.$$
Since Q is injective there is a unique element $\phi(\xi) \in Y$ such that
$$x = x_0 + \xi + Q\phi(\xi).$$
It follows also from the earlier proposition that $\|Q\phi(\xi)\| \le 2c\|f(x_0 + \xi)\| < \delta/2$. We prove that ϕ is of class $\mathscr{C}^\ell$. Define $\Psi : B_\delta(x_0) \to X$ by
$$\Psi(x) := Qf(x) + x - x_0 - QD(x - x_0).$$
Then
$$\Psi(x_0) = 0, \qquad \mathrm{d}\Psi(x) - \mathbb{1} = Q(\mathrm{d}f(x) - D).$$
Hence it follows from Eq. (A.2), that $\|\mathrm{d}\Psi(x) - \mathbb{1}\| \le 1/2$ for every $x \in B_\delta(x_0)$. Hence, by Lemma A.3.2, Ψ has a $\mathscr{C}^\ell$ inverse on $B_{\delta/2}(0)$. For $\xi \in \ker D$ with $\|\xi\| < \delta/8$ we have $\|\xi + Q\phi(\xi)\| < 5\delta/8 < \delta$ and $\Psi(x_0 + \xi + Q\phi(\xi)) = \xi$, and hence
$$x_0 + \xi + Q\phi(\xi) = \psi^{-1}(\xi).$$

Since $DQ = \mathbb{1}$ this implies

$$\phi(\xi) = D\psi^{-1}(\xi) - Dx_0 \qquad \text{for } \xi \in \ker D \text{ with } \|\xi\| < \delta/8.$$

Hence, ϕ is of class $\mathscr{C}^\ell$ as claimed and since $\mathrm{d}\Psi(x_0) = \mathbb{1}$, we have $\mathrm{d}\phi(0) = 0$. Moreover, if $x \in X$ is such that $f(x) = 0$ and $\|x - x_0\| < \delta/8(1 + c\|D\|) < \delta$, then we can write

$$x = x_0 + \xi + Q\eta \qquad \eta := D(x - x_0), \qquad \xi := x - x_0 - Q\eta \in \ker D.$$

Hence we have $\|\eta\| \le \|D\|\|x - x_0\|$ which implies

$$\|\xi\| \le (1 + c\|D\|)\|x - x_0\| < \delta/8.$$

Therefore, $\eta = \phi(\xi)$, so that $x = \psi^{-1}(\xi)$. Thus the map

$$\psi : f^{-1}(0) \cap B_\delta/8(1 + c\|D\|)(x_0) \to \ker D$$

is a coordinate chart on $f^{-1}(0)$. The transition maps are obviously smooth. □

A.4 Sard-Smale theorem

A.4.1 Finite dimensional reduction

We now explain how to obtain a finite dimensional model for the local description of the zero set of a smooth map $f : X \to Y$ between two Banach spaces. Such a finite dimensional reduction is sometimes called a *Kuranishi model*. Assume that $f(0) = 0$ and denote $D := \mathrm{d}f(0) : X \to Y$ .By assumption, D is a bounded linear operator. A **pseudo inverse** of D is a bounded linear operator $T : Y \to X$ which satisfies

$$TDT = T, \qquad DTD = D.$$

The next proposition gives a necessary and sufficient criterion for the existence of a pseudo inverse. It shows, in particular, that every Fredholm operator admits a pseudo inverse.

Proposition A.4.1. A bounded linear operator $D : X \to Y$ admits a pseudo inverse iff D satisfies

1. D has a closed image.
2. The kernel of D has a complement in X.
3. The image of D has a complement in Y.

The proof goes via elementary linear algebra, and helps us prove the following result.

Theorem A.4.2. Let X and Y be Banach spaces, $U \subseteq X$ be an neighborhood of zero, and ℓ be a positive integer. Let $f : U \to Y$ be a $\mathscr{C}^\ell$ map such that $f(0) = 0$ and suppose that the operator $D := \mathrm{d}f(0) : X \to Y$ has a pseudo inverse $T : Y \to X$. Then there is an open neighborhood $W \subseteq X$ of zero, a local $\mathscr{C}^\ell$-diffeomorphism $g : W \to g(W)$ onto an open subset $g(W) \subseteq U$, and a $\mathscr{C}^\ell$ map $f_0 : W \to Y_0 := \ker T$ such that

$$f \circ g(x) = f_0(x) + Dx$$

for $x \in W$ and

$$g(0) = 0, \quad \mathrm{d}g(0) = \mathbb{1}, \quad f_0(0) = 0, \quad \mathrm{d}f_0(0) = 0.$$

Proof. Consider the $\mathscr{C}^\ell$ map $\psi : U \to X$ defined by

$$\psi(x) := x + T(f(x) - Dx).$$

(This is almost the same formula as in the proof of Eq. (A.2) except that the term TDx_1 has been dropped.) This map satisfies

$$\psi(0) = 0, \quad \mathrm{d}\psi(0) = \mathbb{1}.$$

Hence, by the inverse function theorem, ψ has a local inverse defined on some open neighborhood $U_0 \subseteq U$ of zero. Let $W := \psi(U_0)$ and define $g : W \to U_0$ and $f_0 : W \to \ker T$ by

$$g := \psi^{-1}, \qquad f_0 := (\mathbb{1} - DT) \circ f \circ \psi^{-1}.$$

The formula

$$D\psi(x) = Dx + DT(f(x) - Dx) = DTf(x)$$

shows that $D = DT \circ f \circ \psi^{-1}$ and hence

$$f \circ g = f \circ \psi^{-1} = (\mathbb{1} - DT) \circ f \circ \psi^{-1} + D = f_0 + D.$$

□

A.4.2 Sard-Smale theorem

Smale proved the following infinite dimensional version of Sard's theorem.

Theorem A.4.3 (Sard-Smale). Let X and Y be separable Banach spaces and $U \subseteq X$ be an open set. Suppose that $f : U \to Y$ is a Fredholm map of class $\mathscr{C}^\ell$, where

$$\ell \geq \max\{1, \mathrm{ind}(f) + 1\}.$$

Then the set

$$Y_{\mathrm{reg}}(f) := \{y \in Y \mid x \in U, f(x) = y \implies \mathrm{im}\, \mathrm{d}f(x) = Y\}$$

of regular values of f is of the second category in the sense of Baire (a countable intersection of open and dense sets).

Proof. We prove that every point $x_0 \in U$ has an open neighborhood U_0 such that, for every closed subset $V_0 \subseteq U_0$, the set of regular values of the restriction $f|_{V_0}$ is open and dense in Y. Assume without loss of generality that $x_0 = 0 \in U$ and let $T : Y \to X$ be a pseudo inverse of $D := \mathrm{d}f(0)$. Then, by Theorem A.4.2, there is an open neighborhood $Q \subseteq X$ of zero, a $\mathscr{C}^\ell$ diffeomorphism $g : W \to g(W)$ onto an open subset $g(W) \subseteq X$, and a $\mathscr{C}^\ell$ map $f_0 : W \to \ker T$ such that

$$f \circ g = f_0 + D.$$

We claim that $U_0 := g(W)$ is the required neighborhood of $x_0 = 0$.

To see this, recall that there are splittings

$$X = X_0 \oplus X_1, \qquad Y = Y_0 \oplus Y_1,$$

where

$$X_0 := \ker D, \qquad X_1 := \mathrm{im}\, T, \qquad Y_0 := \ker T, \qquad Y_1 := \mathrm{im}\, D.$$

Think of $g : W \to X$ as a coordinate chart on X and write the equation $f(g(x)) = y$ in the form

$$y_0 = f_0(x_0, x_1), \qquad y_1 = D_1 x_1,$$

for $x = x_0 + x_1 \in W$, where $x_i \in X_i$. Here $D_1 : X_1 \to Y_1$ denotes the restriction of D to X_1. It follows from this description that $y = y_0 + y_1$ is a regular value of $f|_{g(W)}$ iff y_0 is a regular value of the map

$$\{x_0 \in X_0 \mid x_0 + D_1^{-1} y_1 \in W\} \to Y_0 : x_0 \mapsto f_0(x_0, D_1^{-1} y_1).$$

Since $\dim X_0 - \dim Y_0 = \mathrm{index}(f)$ and $\ell \geq \max\{1, \mathrm{index}(f) + 1\}$, it follows from Sard's theorem for $\mathscr{C}^\ell$ maps between finite dimensional vector spaces that the set

$$Y_{\mathrm{reg}}(f; g(W)) := \{y \in Y \mid x \ni g(W), f(x) = y \implies \mathrm{im}\, \mathrm{d}f(x) = Y\}$$

of regular values of $f|_{g(W)}$ is dense in Y. Hence $Y_{\mathrm{reg}}(f; g(W)) \subseteq Y_{\mathrm{reg}}(f; g(V))$ is dense for every subset $V \subseteq W$.

We prove that $Y_{\mathrm{reg}}(f; g(V))$ is open whenever $V \subseteq W$ is closed in X. Let

$$y_\nu = y_{\nu,0} + y_{\nu,1} \in Y$$

be a sequence of singular values of $f|_{g(V)}$ which converges to y. Then there exists a sequence $x_\nu \in V$ such that $f(g(x_\nu)) = y_\nu$ and $\mathrm{d}f(g(x_\nu))$ is not surjective. The sequence

$$x_{\nu,1} := TDx_\nu = Ty_\nu$$

converges and, since V is bounded, the sequence

$$x_{\nu,0} := x_\nu - TDx_\nu \in \ker D$$

is bounded. Passing to a subsequence we may assume that $x_{\nu,0}$ converges as well, and hence, so does $x_\nu = x_{\nu,0}+x_{\nu,1}$. Since V is closed, the limit point $x := \lim_{\nu\to\infty} x_\nu$ lies again in V and $f(g(x)) = y$. Moreover, $\mathrm{d}f(g(x))$ is the limit of operators with a nontrivial cokernel and hence cannot be surjective. Hence y is a singular value of $f|_{g(V)}$.

Thus we have proved that every point in U has an open neighborhood U_0 such that the set $Y_{\mathrm{reg}}(f; V_0)$ is open and dense in Y for every closed subset $V_0 \subseteq U_0$. Since X is separable, it follows from the earlier proposition, that the open set U can be covered by countably many such open neighborhoods U_i. Hence U can be covered by countable many closed sets V_i such that $Y_{\mathrm{reg}}(f; V_i)$ is open and dense in Y. Hence

$$Y_{\mathrm{reg}}(f) = \cap_i Y_{\mathrm{reg}}(f; V_i)$$

is a countable intersection of open and dense sets in Y. $\square$

Appendix B

Elliptic regularity

In this appendix, we introduce Sobolev spaces and state the $\mathcal{L}^p$ regularity theory of the Laplace operator which is needed in the theory of J-holomorphic curves. The first section gives a succinct account of Sobolev spaces, with the main embedding results. In the second section, we state the required results concerning the Calderon-Zygmund inequality and the interior regularity for the Laplace operator. All of this is finally used in the last subsection, which explains the elliptic bootstrapping technique for pseudoholomorphic curves.

B.1 Sobolev and Hölder spaces

Weak derivatives. Let $u : \Omega \to \mathbb{R}$ be locally integrable and fix a multi-index $\nu = (\nu_1, \dots, \nu_n)$. A locally integrable function $u_\nu : \Omega \to \mathbb{R}$ is called the *weak derivative* of u corresponding to ν if

$$\int_\Omega u(x)\partial^\nu \phi(x)\,\mathrm{d}x = (-1)^{|\nu|}\int_\Omega u_\nu(x)\phi(x)\,\mathrm{d}x$$

for every test function $\phi \in \mathscr{C}_0^\infty(\Omega)$. The weak derivative, if it exists, is almost everywhere uniquely determined by u and is denoted by $\partial^\nu u := u_\nu$. By the divergence theorem, every $\mathscr{C}^k$ function $u : \Omega \to \mathbb{R}$ has weak derivatives up to order k and these agree with the *strong* (usual) derivatives.

Now fix a non-negative integer k and a number $1 \le p \le \infty$. The **Sobolev space** $\mathcal{W}^{k,p}(\Omega)$ is defined as the space of all (equivalence classes of) functions $u \in \mathcal{L}^p(\Omega)$ such that the weak derivative $\partial^\nu u$ exists and is p-integrable for all ν with $|\nu| \le k$. For $1 \le p < \infty$, define the $\mathcal{W}^{k,p}$-norm of a function $u \in \mathcal{W}^{k,p}(\Omega)$ by

$$\|u\|_{k,p} := \left(\int_\Omega \sum_{|\nu|\le k} |\partial^\nu u(x)|^p\,\mathrm{d}x\right)^{1/p}.$$

The $\mathcal{W}^{k,\infty}$-norm is defined as the maximum of the $\mathcal{L}^\infty$-norms of the weak derivatives $\partial^\nu u$ for $|\nu| \le k$. The space $\mathcal{W}^{k,p}_{\mathrm{loc}}(\Omega)$ is defined as the space of (equivalence classes of) locally p-integrable functions $u : \Omega \to \mathbb{R}$ whose restrictions to all precompact open subsets Q of Ω are in $\mathcal{W}^{k,p}(Q)$. The space $\mathcal{W}^{k,p}_0(\Omega)$ is defined as the closure of $\mathscr{C}_0^\infty(\Omega)$ in $\mathcal{W}^{k,p}(\Omega)$. Thus $\mathcal{W}^{k,p}_0(\Omega)$ is the completion of $\mathscr{C}_0^\infty(\Omega)$ with respect to the $\mathcal{W}^{k,p}$-norm.

Proposition B.1.1. $\mathcal{W}^{k,p}(\Omega)$ is a Banach space, and is reflexive for $1 < p < \infty$. Moreover, $\mathcal{W}^{k,p}_0(\Omega)$ is separable for $1 \le p < \infty$.

Proof. The proof essentially rests on the fact that $\mathcal{L}^p(\Omega)$ is a Banach space, and is reflexive for $1 < p < \infty$. See [7], pp. 166 and 169. The rest of the proof easily follows from the hint given in [2], Ex. B.1.1. □

Approximation by smooth functions. We now state that for a large class of domains $\Omega \subseteq \mathbb{R}^n$, the Sobolev space $\mathcal{W}^{k,p}(\Omega)$ can be identified with the completion of $\mathscr{C}^\infty(\overline{\Omega})$ with respect to the $\mathcal{W}^{k,p}$-norm. The domains we are talking about are called **Lipschitz domains** – open sets whose boundary can locally be represented as the graph of a Lipschitz function (this includes sets with smooth boundaries). Explicitly, this means that for every $x \in \partial\Omega$ there is a neighborhood U of x, a unit vector $\xi \in S^{n-1}$, a constant $\delta > 0$, and a Lipschitz continuous function $f : \xi^\perp \to \mathbb{R}$ such that $f(0) = 0$ and

$$\Omega \cap U = \{x + \eta + t\xi \mid \eta \in \xi^\perp, |\eta| < \delta, f(\eta) < t < \delta\}.$$

The proofs use a very useful technique of smoothing a given function using mollifier functions[1]. It is typically easier to show inequalities for these smooth functions, and then by taking limits, we get the results for arbitrary functions in $\mathcal{W}^{k,p}$. See [2], Appendix 2 for further information.

Some important results about Sobolev spaces which influence our hypotheses for later results include the following.

Sobolev embedding theorems. A function with weak derivatives need not be continuous. Consider for example the function $u(x) = |x|^{-\alpha}$ with $\alpha \in \mathbb{R}$ in the domain $\Omega = B(0,1)$. Then $\partial_j u = -\alpha x_j |x|^{-\alpha-2}$. By induction,

$$|\partial^\nu u(x)| \le c_\nu |x|^{-\alpha-|\nu|}.$$

Now the function $x \mapsto |x|^{-\beta}$ is integrable on B_1 if and only if $\beta < n$. Hence the derivatives of u upto order k will be p-integrable whenever

$$\alpha p + kp < n.$$

If $kp < n$ choose $0 < \alpha < n/p - k$ to obtain a function which is in $\mathcal{W}^{k,p}(B_1)$ but not continuous at 0. For $kp > n$ this construction fails and, in fact, in this case every $\mathcal{W}^{k,p}$-function is continuous:

Theorem B.1.2 (Sobolev embedding)**.** Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain and suppose that $kp > n$ and $0 < \mu = k - n/p < 1$. Then there exists a constant $c = c(k,p,\Omega) > 0$ such that

$$\|u\|_{\mathscr{C}^{0,\mu}} \le c\|u\|_{\mathcal{W}^{k,p}}$$

for $u \in \mathscr{C}^\infty(\bar{\Omega})$. The inclusion $\mathcal{W}^{k,p}(\Omega) \hookrightarrow \mathscr{C}^0(\Omega)$ is compact.

Theorem B.1.3 (Sobolev embedding)**.** Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz domain and suppose that $kp < n$. Then there exists a constant $c = c(k,p,\Omega) > 0$ such that

$$\|u\|_{\mathcal{L}^{np/(n-kp)}} \le c\|u\|_{\mathcal{W}^{k,p}}$$

for $u \in \mathscr{C}^\infty(\bar{\Omega})$. If $q < np/(n-kp)$ then the inclusion $\mathcal{W}^{k,p}(\Omega) \hookrightarrow \mathcal{L}^q(\Omega)$ is compact.

The above are the Sobolev estimates (proofs can be found in [2], B.1.11 and B.1.12). The compactness statements are known as Rellich's theorem. The case $kp = n$ is the borderline situation for these estimates. In this case, the space $\mathcal{W}^{k,p}$ does not embed into the space of continuous functions.

B.1.1 Sections of vector bundles

Let M be an n-dimensional smooth compact manifold and $\pi : E \to M$ be a smooth vector bundle. A section $s : M \to E$ is said to be of class $\mathcal{W}^{k,p}$ if all its local coordinate representations are in $\mathcal{W}^{k,p}$. This definition is independent of the choice of the coordinates. To see this note that if $\phi \in \mathrm{Diff}(\mathbb{R}^n)$, $\Phi \in \mathscr{C}^\infty(\mathbb{R}^n, \mathbb{R}^{N\times N})$ is a matrix-valued function, and $u \in \mathcal{W}^{k,p}_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^N)$, then the composite $\Phi(u \circ \phi)$ belongs to $\mathcal{W}^{k,p}(\Omega)$ for every bounded open set $\Omega \subseteq \mathbb{R}^n$ and there is an estimate

$$\|\Phi(u\circ\phi)\|_{\mathcal{W}^{k,p}(\Omega)} \le c\|u\|_{\mathcal{W}^{k,p}(\Omega)}$$

with $c = c(\phi, \Phi, \Omega)$ independent of u. This holds even when $kp \le n$. To define a norm on the space of $\mathcal{W}^{k,p}$-sections one can take the sum of the $\mathcal{W}^{k,p}$-norms over finitely many charts which cover M.

For functions u with *values* in a manifold the situation is quite difference since one must consider composites of the form $\phi \circ u$, where ϕ is a coordinate change. For such functions it is required that $kp > n$ (see [2] for details). More precisely, let X and M be smooth closed manifolds and suppose that $kp > n = \dim X$. Then the space $X^{k,p} := \mathcal{W}^{k,p}(X, M)$ can be defined as the space of continuous functions $u : X \to M$ which are in local coordinate charts represented by $\mathcal{W}^{k,p}$-functions. This definition can be shown to be independent of the choice of the coordinates. Alternatively, choose an embedding $M \hookrightarrow \mathbb{R}^N$ and define $\mathcal{X}^{k,p}$ as the subset of the space $\mathcal{W}^{k,p}(X, \mathbb{R}^N)$ which consists of functions with values in M. Both these definitions of $\mathcal{X}^{k,p}$ are equivalent. The second definition also applies to the case $kp \le n$: the values of $u \in \mathcal{W}^{k,p}(X, \mathbb{R}^N)$ are required to be in M almost everywhere. However, in this case the space $\mathcal{X}^{k,p}$ depends on the choice of the embedding.

[1] Intuitively, a mollifier is a function f that you convolve with another function g to get a function which is "close" to g but "nicer". For instance, g might be a general L^1 function and $g * f$ might be a smooth, compactly supported approximation to g. Really a mollifier is not one function but a sequence, or even sometimes a one-parameter continuous family. This contrasts the notion of a cutoff function which is usually smooth, 1 on some set K of interest, and 0 outside a slightly larger set A of interest. Cutoff functions are commonly constructed as the mollification of an indicator function.

B.2 Elliptic bootstrapping

B.2.1 Calderon Zygmund inequality

The Laplace operator. Denote by

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

the Laplace operator on $\mathbb{R}^n$. A $\mathscr{C}^2$-function $u : \Omega \to \mathbb{R}$ on an open set $\Omega \subseteq \mathbb{R}^n$ is called **harmonic** if $\Delta u = 0$. Harmonic functions are real analytic. (If $n = 2$ then a function is harmonic iff it is locally the real part of a holomorphic function.) Harmonic functions are characterized by the **mean value property**

$$u(x) = \frac{n}{\omega_n r^2} \int_{B_r(x)} u(\xi)\, \mathrm{d}\xi, \qquad B_r(x) \subseteq \Omega.$$

Here $\omega_n = 2\pi^{n/2}\Gamma(n/2)^{-1}$ is the volume of the unit sphere in $\mathbb{R}^n$. In particular, $\omega_2 = 2\pi$.

The **fundamental solution** of Laplace's equation is the function

$$K(x) := \begin{cases} (2\pi)^{-1} \log|x| & n = 2 \\ (2-n)^{-1}\omega_n^{-1}|x|^{2-n}, & n \geq 3. \end{cases}$$

Formulae for its first and second derivatives ensure that $\Delta K = 0$. In fact, every compactly supported $\mathscr{C}^2$ function $u : \mathbb{R}^n \to \mathbb{R}$ satisfies

$$u = K * \Delta u, \qquad \partial_j u = K_j * \Delta u,$$

where $*$ denotes convolution, and $K_j := \partial K / \partial x_j$. Conversely,

$$\Delta(K * f) = f, \qquad \Delta(K_j * f) = \partial_j f$$

for every $f \in \mathscr{C}_0^\infty(\mathbb{R}^n)$. This is **Poisson's identity**. In general $K * f$ will not have compact support. Since the second derivatives of K are not integrable on compact sets there exists a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ such that $K * f \notin \mathscr{C}^2$. For such a function f there is no classical solution of $\Delta u = f$. The situation is however quite different for weak solutions. Let $f \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$. A function $u \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$ is called a **weak solution** of $\Delta u = f$ if

$$\int_\Omega u(x)\Delta\phi(x)\, \mathrm{d}x = \int_\Omega f(x)\phi(x)\, \mathrm{d}x$$

for $\phi \in \mathscr{C}_0^\infty(\Omega)$. Similarly, $u \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$ is called a weak solution of $\Delta u = \partial_j f$ with $f \in L^1_{\mathrm{loc}}$ if

$$\int_\Omega u(x)\Delta\phi(x)\, \mathrm{d}x = -\int_\Omega f(x)\partial_j\phi(x)\, \mathrm{d}x$$

for $\phi \in \mathscr{C}_0^\infty(\Omega)$.

Lemma B.2.1. Let $u, f \in L^1(\mathbb{R}^n)$ with compact support.

1. u is a weak solution of $\Delta u = f$ iff $u = K * f$.
2. u is a weak solution of $\Delta u = \partial_j f$ iff $u = K_j * f$.

See [2], Lemma B.2.2 for a proof of the above. We now state the Calderon Zygmund inequality:

Theorem B.2.2 (Calderon-Zygmund inequality)**.** Let K be the fundamental solution of Laplace's equation on $\mathbb{R}^n$, and suppose that $1 < p < \infty$. Then there exists a constant $c = c(n, p) > 0$ such that

$$\|\nabla(K_j * f)\|_{L^p} \leq c\|f\|_{L^p}$$

for every $u \in \mathscr{C}_0^\infty(\mathbb{R}^n)$.

The above inequality is the fundamental estimate for the $\mathcal{L}^p$ theory of elliptic operators. The proof is full of subtle tricks, and can be found in [2], Theorem B.2.7.

B.2.2 Regularity for the Laplace operator

We shall now state the crucial regularity theorems and estimates for the Laplace operator.

Theorem B.2.3 (Interior regularity). Let $1 < p < \infty$, $k \geq 0$ be an integer, and $\Omega \subseteq \mathbb{R}^n$ be an open domain. If $u \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$ is a weak solution of

$$\Delta u = f, \qquad f \in \mathcal{W}^{k,p}_{\mathrm{loc}}(\Omega),$$

then $u \in W^{k+2,p}_{\mathrm{loc}}(\Omega)$. Moreover, for every bounded open set $\Omega' \subseteq \mathbb{R}^n$ with $\overline{\Omega'} \subseteq \Omega$ there is a constant $c = c(k,p,n,\Omega',\Omega) > 0$ such that, for every $u \in \mathscr{C}^\infty(\bar{(\Omega)})$,

$$\|u\|_{\mathcal{W}^{k+2,p}(\Omega')} \leq c\left(\|\Delta u\|_{\mathcal{W}^{k,p}(\Omega)} + \|u\|_{\mathcal{L}^p(\Omega)}\right).$$

Sometimes it is useful to consider weak solutions of $\Delta u = f$ where f is not a function but a distribution in $W^{-1,p}$. We rephrase this in terms of weak solutions of the equation $\Delta u = \operatorname{div} f$ with $f \in \mathcal{L}^p$.

Theorem B.2.4. Let $1 < p < \infty$ and $\Omega' \subseteq \Omega \subseteq \mathbb{R}^n$ be open sets such that $\overline{\Omega'} \subseteq \Omega$. Then there exists a constant $c = c(p,n,\Omega',\Omega) > 0$ such that the following holds. Assume that $u \in \mathcal{L}^1_{\mathrm{loc}}(\Omega)$ and $f = (f_0, \ldots, f_n) \in \mathcal{L}^p_{\mathrm{loc}}(\Omega, \mathbb{R}^{n+1})$ satisfy

$$\int_\Omega u(x)\Delta\phi(x)\,\mathrm{d}x = \int_\Omega f_0(x)\phi(x)\,\mathrm{d}x - \sum_{j=1}^n \int_\Omega f_j(x)\partial_j\phi(x)\,\mathrm{d}x$$

for every $\phi \in \mathscr{C}^\infty_0(\Omega)$. Then $u \in \mathcal{W}^{1,p}_{\mathrm{loc}}(\Omega)$ and

$$\|u\|_{\mathcal{W}^{1,p}(\Omega')} \leq c\left(\|f\|_{\mathcal{L}^p(\Omega)} + \|u\|_{\mathcal{L}^p(\Omega)}\right).$$

The proofs of the above results involve heavy analysis and can be found in [2], Section B.3.

B.2.3 Elliptic bootstrapping

Let $\mathcal{J}^\ell(M)$ be the space of $\mathscr{C}^\ell$ almost complex structures ($1 \leq \ell \leq \infty$) on M^{2n}, and $\mathcal{J}(\Sigma)$ be the space of complex structures on an oriented surface Σ.

Theorem B.2.5 (Regularity). Fix $\ell \in \mathbb{N}$ and $p > 2$. Let $j \in \mathcal{J}(\Sigma)$, $J \in \mathcal{J}^\ell(M)$ and $u : \Sigma \to M$ be a $\mathcal{W}^{1,p}$ function such that $\mathrm{d}u \circ j = J \circ \mathrm{d}u$. Then u is of class $\mathcal{W}^{\ell+1,p}_{\mathrm{loc}}$. If $\ell = \infty$, then u is smooth.

Theorem B.2.6 (Compactness). Fix $\ell \in \mathbb{N}$ and $p > 2$. Let $J_\nu \in \mathcal{J}^\ell(M)$ be a sequence converging to J in the $\mathscr{C}^\ell$ topology, $j_\nu \in \mathcal{J}(\Sigma)$ a sequence converging to j in $\mathscr{C}^\infty$-topology. Let $U_\nu \subseteq \Sigma$ be an increasing sequence of open sets whose union is Σ and $u_\nu : U_\nu \to M$ be a sequence of (j_ν, J_ν)-holomorphic curves of class $\mathcal{W}^{1,p}$. Assume that for all compact sets $Q \subseteq \Sigma$ (with smooth boundary[2]), there exists a compact set $K \subseteq M$ and a constant $c > 0$ such that

$$\|\mathrm{d}u_\nu\|_{\mathcal{L}^p(Q)} \leq c, \qquad u_\nu(Q) \subseteq K,$$

for ν sufficiently large. Then there exists a subsequence of u_ν converging in $\mathscr{C}^\ell$-topology on every compact subset of Σ.

Both these theorems are obvious when J is integrable and Σ is closed because each component of a holomorphic curve in holomorphic coordinates is a harmonic function. In particular, the compactness theorem follows from the mean value property of harmonic functions.

The main idea of the proof of both theorems is to replace the nonlinear equation $\partial_s u + J(u)\partial_t u = 0$ by the linear equation $\partial_s u + J'\partial_t u = 0$, where $J'(z) = J(u(z))$ is a matrix valued function on some open set $\Omega \subseteq \mathbb{C}$. Note that J' will have the same smoothness as u and this leads to the elliptic bootstrapping argument explained below.

More precisely, suppose $J \in \mathcal{W}^{1,p}(\Omega, \mathbb{R}^{2n\times 2n})$ such that $J^2 = -\mathbb{1}$. Let $u \in \mathcal{W}^{1,p}(\Omega, \mathbb{R}^{2n})$ be a solution to $\partial_s u + J\partial_t u = 0$. Then the derivative $v := \partial_s u = -J\partial_t u$ solves the inhomogeneous equation

$$\partial_s v + J\partial_t v = \partial_s(-J\partial_t u) + J(\partial_s\partial_t u) = -(\partial_s J)\partial_t u \quad =: \eta \qquad \text{(by Leibnitz' rule)}.$$

[2] That is, for density of smooth functions in $\mathcal{W}^{k,p}$.

The best we can expect from our regularity theorems is the statement that if $\eta \in \mathcal{L}^r(\Omega)$ for some $r > 1$ then v will belong to $\mathcal{W}^{1,r}(\Omega)$. We would like to apply this with $r = p$. However, our assumptions imply that the functions $\partial_s J$ and $\partial_t u$ are in $\mathcal{L}^p$ and so their product in general only belong to $\mathcal{L}^{p/2}$ (by Holder). To deal with this we improve the regularity of u while that of $\partial_s J$ is fixed. Thus suppose we have already proved that $\partial_t u$ belongs to $\mathcal{L}^q$ for some $q > p/(p-1)$. Then η belongs to $\mathcal{L}^r$ where $1/p + 1/q = 1/r$. Since $r > 1$ we then find that $v \in \mathcal{W}^{1,r}$ and hence $u \in \mathcal{W}^{2,r}$. If $1 < r < 2$, then by Theorem B.1.3, $u \in \mathcal{W}^{1,q'}$, where $q' := 2r(2-r) > q$. Thus, we have gained something. Now repeat this process. Once we get to a value for r that is greater than 2 we obtain from Theorem B.1.2 that $\partial_t u$ is continuous and η belongs to $\mathcal{L}^p$. This iteration is part of the *elliptic bootstrapping* argument. It requires the following two lemmas.

Lemma B.2.7. Given $p > 2$ there exists a finite increasing sequence $q_0 < q_1 < \cdots < q_m$ such that, for $j = 0, \ldots, m-1$, we have

$$\frac{p}{p-1} < q_0 \le p, \qquad q_{m-1} < \frac{2p}{p-2} < q_m,$$

$$q_{j+1} = \frac{2r_j}{2-r_j}, \qquad r_j := \frac{pq_j}{p+q_j}.$$

Proof. Consider the map $h : (p/(p-1), 2p/(p-2)) \to (2, \infty)$ defined by

$$h(q) := \frac{2pq}{2p+2q-pq} = \frac{2r}{2-r}, \qquad r := \frac{pq}{p+q} < 2.$$

The condition $r < 2$ is equivalent to $q < 2p/(p-2)$. The map h is a monotonically increasing diffeomorphism such that $h(q) > q$. Now choose the sequence q_j such that $q_{j+1} = h(q_j)$. □

Lemma B.2.8. Assume $p > 2$ and $1 < r \le p$. Then there exists a constant $c = c(p,r)$ such that the following holds. If $f \in \mathcal{W}^{1,p}(\mathbb{R}^2)$ and $g \in \mathcal{W}^{1,r}(\mathbb{R}^2)$, then $fg \in \mathcal{W}^{1,r}(\mathbb{R}^2)$ and

$$\|fg\|_{\mathcal{W}^{1,r}} \le c\|f\|_{\mathcal{W}^{1,p}}\|g\|_{\mathcal{W}^{1,r}}.$$

Proof. Examine the $\mathcal{L}^r$-norm of the expression $\mathrm{d}(fg) = (\mathrm{d}f)g + f(\mathrm{d}g)$. Then the term $f(\mathrm{d}g)$ can be estimated by the sup norm of f and the $\mathcal{W}^{1,r}$-norm of g. The term $(\mathrm{d}f)g$ can be estimated by

$$\|(\mathrm{d}f)g\|_{\mathcal{L}^r} \le \|\mathrm{d}f\|_{\mathcal{L}^p}\|g\|_{\mathcal{L}^q}, \qquad 1/p + 1/q = 1/r.$$

If $r < 2$ then, since $p > 2$, we have $q = pr/(p-r) < 2r/(2-r)$ and the $\mathcal{L}^q$ norm of g can be estimated by the $\mathcal{W}^{1,r}$-norm. The latter is obvious when $r \ge 2$. □

We next prove a local estimate for the solutions of the linear Cauchy-Riemann equation. Let $\Omega \subseteq \mathbb{C}$ be open. Let $u : \Omega \to \mathbb{R}^{2n}$ be a solution to $\partial_s u + J\partial_t u = \eta$, where $\eta : \Omega \to \mathbb{R}^{2n}$ and $J : \Omega \to \mathbb{R}^{2n\times 2n}$ satisfies $J^2 = -\mathbb{1}$. We shall first prove regularity of u under the weakest possible regularity assumptions on J and η. Let's assume that J is of class $\mathcal{W}^{1,p}$. If u is a solution to the perturbed CR equation (as above), then by partial integration,

$$\int_\Omega \langle \partial_s\phi + J^T\partial_t\phi, u\rangle = -\int_\Omega \langle \phi, \eta + (\partial_t J)u\rangle \tag{B.1}$$

for every test function $\phi \in \mathscr{C}_0^\infty(\Omega, \mathbb{R}^{2n})$, where J^T denotes the transpose of J. The next lemma asserts the converse.

Lemma B.2.9. Let $\Omega' \subseteq \Omega \subseteq \mathbb{C}$ be open sets such that $\overline{\Omega'} \subseteq \Omega$ and p, q, r be positive real numbers (including possibly plus infinity, but not all infinite at once) such that

$$2 < p, \qquad 1 < r < \infty, \qquad 1/p + 1/q = 1/r.$$

Then for every constant $c_0 > 0$ there exists a constant $c > 0$ with the following significance. Assume $J \in \mathcal{W}^{1,p}(\Omega, \mathbb{R}^{2n\times 2n})$ satisfies $J^2 = -\mathbb{1}$ and $\|J\|_{\mathcal{W}^{1,p}(\Omega)} \le c_0$. Then the following holds.

1. If $u \in \mathcal{L}^q_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n})$ and $\eta \in \mathcal{L}^r_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n})$ are such that Eq. (B.1) holds, then u satisfies $\partial_s u + J\partial_t u = \eta$ almost everywhere.

2. If $u \in \mathcal{W}^{1,r}_{\mathrm{loc}}(\Omega)$, then
$$\|u\|_{\mathcal{W}^{1,r}(\Omega')} \leq c\left(\|\partial_s u + J\partial_t u\|_{\mathcal{L}^r(\Omega)} + \|u\|_{\mathcal{L}^q(\Omega)}\right).$$

The proof is worked out in [2], pp. 535. Another important proposition is the following, whose proof can be found in [2], pp. 537.

Proposition B.2.10. Let $\Omega' \Subset \Omega \Subset \mathbb{C}$ be open sets such that $\overline{\Omega'} \subseteq \Omega$, ℓ be a positive integer, and $p > 2$. Then for every constant $c_0 > 0$ there exists a constant $c > 0$ with the following significance. Assume $J \in \mathcal{W}^{\ell,p}(\Omega, \mathbb{R}^{2n\times 2n})$ satisfies $J^2 = -\mathbb{1}$ and
$$\|J\|_{\mathcal{W}^{\ell,p}(\Omega)} \leq c_0.$$
Then the following holds for every $k \in \{0, \dots, \ell\}$. If $u \in \mathcal{L}^p_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n})$ and $\eta \in \mathcal{W}^{k,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n})$ are such that Eq. (B.1) holds then $u \in \mathcal{W}^{k+1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n})$ and u satisfies $\partial_s u + J\partial_t u = \eta$ almost everywhere.

Proof (of Theorem B.2.5). It suffices to prove the result in local holomorphic coordinates on Σ and in local coordinates on M. Pushing J forward with a $\mathscr{C}^\ell$ coordinate chart gives a $\mathscr{C}^{\ell-1}$ almost complex structure on $\mathbb{R}^{2n}$ (since the push forward is by conjugation with the derivative of the coordinate function). Hence we shall assume that $\Omega \subseteq \mathbb{C}$ and $u : \Omega \to \mathbb{R}^{2n}$ is a solution to $\partial_s u + J(u)\partial_t u = 0$, where $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n\times 2n}$ is of class $\mathscr{C}^{\ell-1}$. Now first assume that $k = 1$. Then $u \in \mathcal{W}^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n})$ satisfies the requirements of Proposition B.2.10 with $k = 1, \eta = 0$, and J replaced by $J \circ u \in \mathcal{W}^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{2n\times 2n})$ for $k = 1, \dots, \ell - 1$. Hence $u \in \mathcal{W}^{\ell,p}_{\mathrm{loc}}$. □

Proof (of Theorem B.2.6). Since the inclusion $\mathcal{W}^{1,p} \hookrightarrow \mathscr{C}^0$ is compact for $p > 2$, by passing to a subsequence, we may assume that u_ν converges uniformly to a continuous function $u : \Sigma \to M$. Hence, we may assume that $u_\nu : \Omega \to \mathbb{R}^{2n}$ is a sequence of $\mathcal{W}^{1,p}_{\mathrm{loc}}$ solutions of the sequence of problems
$$\partial_s u_\nu + J_\nu(u_\nu)\partial_t u_\nu = 0,$$
where $J_\nu \in \mathscr{C}^{\ell-1}(\mathbb{R}^{2n}, \mathbb{R}^{2n\times 2n})$ converges to J in the $\mathscr{C}^{\ell-1}$-topology. It follows from Proposition B.2.10 by induction that $u_\nu \in \mathcal{W}^{\ell,p}_{\mathrm{loc}}$ and
$$\sup_\nu \|u_\nu\|_{\mathcal{W}^{\ell,p}(Q)} < \infty$$
for every compact subset $Q \subseteq \Omega$ with smooth boundary. By Theorem B.1.2, the inclusion $\mathcal{W}^{\ell,p}(Q) \hookrightarrow \mathscr{C}^{\ell-1}(Q)$ is compact. This proves the existence of a subsequence which converges in the $\mathscr{C}^{\ell-1}$-topology on any given compact subset of Σ. The theorem follows by choosing a diagonal subsequence associated to an exhausting sequence of compact subsets of Σ. □

Appendix C

The Riemann Roch theorem

This appendix gives an introduction to real Cauchy-Riemann operators on closed Riemann surfaces and states the Riemann-Roch theorem in this case. The first section discusses the basic background material and states the Riemann-Roch theorem in the full generality. In the second section, we state the main results required to prove that CR operators are Fredholm. Finally, the last section proves a special case of Riemann Roch theorem, for bundles on S^2.

C.1 Cauchy Riemann operators

Let (Σ, j) be a closed Riemann surface and $(E, J) \to \Sigma$ be a (smooth) complex vector bundle. We denote by $\Omega^k(\Sigma, E)$ the smooth E-valued k-forms on Σ and by $\Omega^{p,q}(\Sigma, E)$ the forms of type (p,q). The complex structure j on Σ determines a $\mathbb{C}^*$ action on $\Omega^k(\Sigma, E)$ via $(a+ib)\cdot\alpha := a\alpha + bj^*\alpha$; the direct sum decomposition $\Omega^k(\Sigma, E) = \oplus_{p+q=k}\Omega^{p,q}(\Sigma, E)$ are the isotypic components of this action. Replacing j by $-j$ interchanges $\Omega^{p,q}$ and $\Omega^{q,p}$. We omit E when we are talking about $E = TM$.

Denote by $\overline{\partial} : \Omega^0(\Sigma) \to \Omega^{0,1}(\Sigma)$ the composition of the exterior derivative $d : \Omega^0(\Sigma) \to \Omega^1(\Sigma)$ with the projection $\Omega^1(\Sigma) = \Omega^{1,0}(\Sigma) \oplus \Omega^{0,1}(\Sigma) \to \Omega^{0,1}(\Sigma)$. A (complex linear, smooth) **Cauchy-Riemann operator** on the bundle $E \to \Sigma$ is a $\mathbb{C}$-linear operator $D : \Omega^0(\Sigma, E) \to \Omega^{0,1}(\Sigma, E)$ which satisfies the Leibniz rule

$$D(f\xi) = f(D\xi) + (\overline{\partial} f)\xi$$

for $\xi \in \Omega^0(\Sigma, E)$ and $f \in \Omega^0(\Sigma)$. For example, $\overline{\partial}$ (acting coordinate wise) is a Cauchy-Riemann operator on the trivial bundle. A Cauchy-Riemann operator on E extends uniquely to a skew wedge derivation $D : \Omega^{p,q}(\Sigma, E) \to \Omega^{p,q+1}(\Sigma, E)$ (same name, different input) satisfying the same Leibniz rule.

Remark C.1.1. A Hermitian structure on E is a real linear product $\langle \cdot, \cdot \rangle$ on E such that the complex structure J is orthogonal. A **Hermitian connection** on E is a $\mathbb{C}$-linear operator $\nabla : \Omega^0(\Sigma, E) \to \Omega^1(\Sigma.E)$ such that

$$\nabla(f\xi) = f\nabla\xi + (\mathrm{d}f)\xi, \quad \mathrm{d}\langle \xi_1, \xi_2 \rangle = \langle \nabla\xi_1, \xi_2 \rangle + \langle \xi_1, \nabla\xi_2 \rangle$$

for all $f \in \Omega^0(\Sigma)$ and $\xi, \xi_1, \xi_2 \in \Omega^0(\Sigma, E)$. Every Hermitian connection ∇ determines a Cauchy-Riemann operator $\overline{\partial}^{\nabla}$ on E defined by

$$\overline{\partial}^{\nabla}\xi := (\nabla\xi)^{0,1} = (\nabla\xi + J\nabla\xi \circ j)/2$$

for $\xi \in \Omega^0(\Sigma, E)$.

Fix a Hermitian structure on E, a Hermitian connection ∇ and a volume form dvol on Σ which is compatible with the orientation determined by j (in the sense that $\mathrm{dvol}(v, jv) > 0$ for $0 \neq v \in T\Sigma$). The volume form and Hermitian structure determine $\mathcal{L}^2$ inner products on the spaces $\Omega^k(\Sigma, E)$. The *formal adjoint* operator of $\overline{\partial}^{\nabla}$ is an operator characterized by the identity

$$\int_\Sigma \left\langle \left(\overline{\partial}^{\nabla}\right)^* \eta, \xi \right\rangle \mathrm{dvol} = \int_\Sigma \left\langle \eta, \overline{\partial}^{\nabla} \xi \right\rangle \mathrm{dvol}$$

for $\xi \in \Omega^0(\Sigma, E)$ and $\eta \in \Omega^{0,1}(\Sigma, E)$. It is called the *formal* adjoint, since it is defined only on those $\eta \in \Omega^{0,1}(\Sigma, E)$ for which the right hand side of the above equality makes sense for all $\xi \in \Omega^0(\Sigma, E)$ (in general, this can be a much smaller domain than all of $\Omega^{0,1}(\Sigma, E)$ and typically contains only compactly supported $(0,1)$-forms with values in E). For the same reasons, the term "formal" is used for the adjoints of general partial differential operators.

Real linear Cauchy-Riemann operators. For our applications, we need to consider more general Cauchy-Riemann operators whose zeroth order term is $\mathbb{R}$-linear rather than $\mathbb{C}$-linear. It will be useful to weaken our smoothness conditions to deal with Banach spaces. Consider the Sobolev spaces $\mathcal{W}^{k,p}(\Sigma, E)$ and $\mathcal{W}^{k,p}(\Sigma, \Lambda^{0,1}T^*\Sigma \otimes E)$.

Definition C.1.2. Let k be a positive integer and $p > 1$ such that $kp > 2$. A **real linear Cauchy-Riemann operator of class** $\mathcal{W}^{k-1,p}$ on E is an operator of the form $D = D_0 + \alpha$, where $\alpha \in \mathcal{W}^{k-1,p}(\Sigma, \Lambda^{0,1}T^*\Sigma \otimes \mathrm{End}_\rho(E))$ and D_0 is a smooth complex linear Cauchy-Riemann operator on E.

Real linear Cauchy-Riemann operators satisfy the equation

$$D(f\xi) = f(D\xi) + (\overline{\partial} f)\xi$$

only for real valued functions f. As in the complex case, they arise via the formula

$$\overline{\partial}^{\nabla}\xi := (\nabla\xi)^{0,1} = \frac{1}{2}(\nabla\xi + J\nabla\xi \circ j)$$

from connections on (E, J). Here, the connection potentials need only to be of class $\mathcal{W}^{k-1,p}$. More precisely, let ∇_0 be any smooth Hermitian connection on E. Then we may write

$$\nabla = \nabla_0 + A, \qquad A \in \mathcal{W}^{k-1,p}(\Sigma, T^*\Sigma \otimes_\rho \mathrm{End}_{\mathbb{R}}(E)),$$

so that $D = \overline{\partial}^{\nabla}$ has the form

$$D\xi = D_0\xi + (A\xi)^{0,1},$$

where $D_0 = \overline{\partial}^{\nabla_0}$ is the (complex linear) Cauchy-Riemann operator associated to ∇_0.

The Riemann-Roch theorem. A Riemann-Roch problem on a vector bundle $E \to \Sigma$ is an operator

$$D : \mathcal{W}^{k,p}(\Sigma, E) \to \mathcal{W}^{k-1,p}(\Sigma, \Lambda^{0,1}T^*\Sigma \otimes E).$$

As always, we assume $p > 1$ and $kp > 2$. Here is the main theorem of this appendix.

Theorem C.1.3 (Riemann Roch). Let $E \to \Sigma$ be a complex vector bundle over a closed Riemann surface. Let D be a real linear Cauchy-Riemann operator on E of class $\mathcal{W}^{k-1,p}$, where $p > 1$ and $kp > 2$. Then the following holds for every integer $\ell \in \{1, \dots, k\}$ and every $q > 1$ such that $\ell - 2/q \le k - 2/p$.

1. The operator D and its formal adjoint D^* are Fredholm. Moreover, their kernels are independent of ℓ and q, and we have

$$\eta \in \operatorname{im} D \iff \int_\Sigma \langle \eta, \eta_0 \rangle \, \mathrm{d}\,\mathrm{vol} = 0 \;\forall\; \eta_0 \in \ker D^*$$

$$\xi \in \operatorname{im} D^* \iff \int_\Sigma \langle \xi, \xi_0 \rangle \, \mathrm{d}\,\mathrm{vol} = 0 \;\forall\; \xi \in \ker D$$

2. The real Fredholm index of D is given by

$$\operatorname{ind}(D) = n\chi(\Sigma) + 2\langle c_1(E), [\Sigma] \rangle$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ, and $c_1(E)$ denotes the first Chern class of (E, J).

Remark C.1.4. The Serre duality theorem asserts that for a closed, complex manifold of dimension n and a holomorphic vector bundle $E \to M$ there is an isomorphism

$$H^q(M, \Omega^p(E^*)) \simeq H^{n-q}(M, \Omega^{n-p}(E))^*$$

of complex vector spaces, where $E^* \simeq (E, -J)$ denotes the dual bundle. These are sheaf cohomology groups and $\Omega^p(E)$ denotes the sheaf of germs of holomorphic p-forms on M with values in E. If $M = \Sigma$ is a closed Riemann surface and $p + q = 1$, this duality statement reduces to assertion (1) above applied to the usual delbar operator $D = \overline{\partial}$. For in this case, $H^0(\Sigma, \Omega^1(E^*))$ is the space of holomorphic sections of the bundle of holomorphic 1-forms $\Omega^1(E^*)$ and can be identified with $\ker \overline{\partial}^*$, while $H^1(\Sigma, \Omega^0(E))$ can be identified with the Dolbeault cohomology group $H^1_{\overline{\partial}}(\Sigma, E)$ which is just $\operatorname{coker} \overline{\partial}$.

C.2 Elliptic estimates

We shall now state the Fredholm property for Cauchy-Riemann operators and outline the duality statement of Theorem C.1.3. The proof of the Fredholm property is based on Lemma A.2.1 which states that an operator D has closed image and finite dimensional kernel provided that a certain estimate holds. The required estimate for smooth Cauchy-Riemann operators is stated in Lemma C.2.1 below. Hence in the smooth case the operators D and D^* have finite dimensional kernel and closed image. The same result holds in general because, as we show in Lemma C.2.2 below, every Cauchy-Riemann operator of class $\mathcal{W}^{\ell-1,p}$ is a compact perturbation of a smooth Cauchy Riemann operator.

It remains to prove that the co-kernels are finite dimensional as well. This follows from a duality theorem which asserts that the cokernel of D can be identified with the kernel of D^* and vice versa. A refined version of this duality statement is formulated in below. The proof is based on elliptic regularity and requires a number of steps under our weak regularity assumptions on the coefficients of the operator D. However, the smooth case is a straightforward consequence of the regularity results in the Section B.2.

Throughout this section we assume that $(\Sigma, j, \mathrm{d\,vol})$ is a compact Riemann surface, that E is a complex vector bundle over Σ, and $\langle\cdot,\cdot\rangle$ is a Hermitian inner product on E.

Lemma C.2.1. Let D be a smooth real linear Cauchy-Riemann operator. Then, for every positive integer k and every $q > 1$ there exists a constant $c > 0$ such that

$$\|\xi\|_{\mathcal{W}^{k,p}} \leq c(\|D\xi\|_{\mathcal{W}^{k-1,q}} + \|\xi\|_{\mathcal{W}^{k-1,q}})$$

for every $\xi \in W^{k,q}(\Sigma, E)$ and

$$\|\eta\|_{\mathcal{W}^{k,q}} \leq c(\|D^*\eta\|_{\mathcal{W}^{k-1,q}} + \|\eta\|_{\mathcal{W}^{k-1,q}}\|$$

for every $\eta \in \mathcal{W}^{k,q}(\Sigma, \Lambda^{0,1}T^*\Sigma \otimes_{\mathbb{C}} E)$.

It follows from Lemma A.2.1 and Lemma C.2.1 that, in the smooth case, both operators D and D^* have a finite dimensional kernel and a closed image. That this continues to hold for Cauchy-Riemann operators of class $\mathcal{W}^{k-1,p}$ is a consequence of Corollary A.2.2 and the next lemma.

Lemma C.2.2. Let ℓ be an integer and $p > 1$ such that $\ell p > 2$. Let $A \in \mathcal{W}^{\ell-1,p}(\Sigma, T^*\Sigma \otimes_{\mathbb{R}} \mathrm{End}_{\mathbb{R}}(E))$. Then the linear operator

$$W^{k,q}(\Sigma, E) \to W^{k-1,q}(\Sigma, \Lambda^{0,1}T^*\Sigma \otimes E) : \xi \mapsto (A\xi)^{0,1}$$

is well defined and compact whenever $1 \leq k \leq \ell$ and $q > 1$ and $k - 2/q \leq \ell - 2/p$.

Proof. If $k = \ell$ then $q \leq p$ and hence, by Holder's inequality $\mathcal{W}^{\ell-1,p} \subseteq \mathcal{W}^{k-1,q}$. If $k \leq \ell - 2$ then, by Theorem B.1.2 and Theorem B.1.3, we obtain $\mathcal{W}^{\ell-1,p} \subseteq \mathscr{C}^{k-1} \subseteq \mathcal{W}^{k-1,q}$ for every q. If $k = \ell - 1$, then the inequality $k - 2/q \leq \ell - 2/p$ is equivalent to $2/p - 1 \leq 2/q$ and hence, in the case $p < 2$, to $q \leq 2p/(2-p)$. Hence, by Theorem B.1.3, we have $\mathcal{W}^{\ell-1,p} \subseteq \mathcal{W}^{k-1,q}$. This holds in all cases. Thus, we have proved that $A \in \mathcal{W}^{k-1,q}$.

Now suppose that $kq > 2$. Then the map $\mathcal{W}^{k,q} \to \mathcal{W}^{k-1,q} : \xi \mapsto (A\xi)^{0,1}$ factors through the inclusion $\mathcal{W}^{k,q} \hookrightarrow \mathscr{C}^{k-1}$. By Theorem B.1.2, this inclusion is compact. This proves the lemma in the case $kq > 2$.

If $kq \leq 2$ we have $k = 1$ and $q \leq 2$. Moreover, by Theorem B.1.3, we have $A \in L^s$ for some $s > 2$. Let $r := sq/(s-q) < 2q/(2-q)$. Then, by Theorem B.1.3, the inclusion $\mathcal{W}^{1,q} \in L^r$ is compact. Moreover, since $1/r + 1/s = 1/q$, it follows from Holder's inequality that multiplication by A defines a bounded linear operator from L^r to L^q. This proves the lemma in the case $kq \leq 2$. □

To establish the Fredholm property of Cauchy-Riemann operators we must prove that the cokernel of D can be identified with the kernel of D^*. This is the content of the following theorem which restates assertion (i) of Theorem C.1.3.

Theorem C.2.3. Let D be a real linear Cauchy-Riemann operator of class $\mathcal{W}^{\ell-1,p}$, where ℓ is a positive integer and $p > 1$ such that $\ell p > 2$. Let k be an integer such that $1 \leq k \leq \ell$ and $q, r > 1$ such that $2/r - 1 \leq k - 2/q \leq \ell - 2/p$. Then the following holds.

1. The operators
$$D : \mathcal{W}^{k,q}(\Sigma, E) \to \mathcal{W}^{k-1,q}(\Sigma, \Lambda^{0,1}T^*\Sigma \otimes_{\mathbb{C}} E), D^* : \mathcal{W}^{k,q}(\Sigma, \Lambda^{0,1}T^*\Sigma \otimes_{\mathbb{C}} E) \to \mathcal{W}^{k-1,q}(\Sigma, E)$$
are Fredholm and $\mathrm{ind}(D) + \mathrm{ind}(D^*) = 0$.
2. If $\eta \in L^r(\Sigma, \Lambda^{0,1}T^*\Sigma \otimes_{\mathbb{C}} E)$ and $\xi \in \mathcal{W}^{k-1,q}(\Sigma, \Lambda^{0,1}T^*\Sigma \otimes_{\mathbb{C}} E)$ satisfy
$$\int_\Sigma \langle \eta, D\zeta \rangle \,\mathrm{d\,vol} = \int_\Sigma \langle \xi, \zeta \rangle \,\mathrm{d\,vol}$$
for every $\zeta \in W^{k,q}(\Sigma, E)$ then $\eta \in W^{k,q}(\Sigma, \Lambda^{0,1}T^*\Sigma \otimes_{\mathbb{C}} E)$ and $D^*\eta = \xi$.
3. If $\xi \in L^r(\Sigma, E)$ and $\eta \in \mathcal{W}^{k-1,q}(\Sigma, \Lambda^{0,1}T^*\Sigma \otimes_{\mathbb{C}} E)$ satisfy
$$\int_\Sigma \langle D^*\zeta, \xi \rangle \,\mathrm{d\,vol} = \int_\Sigma \langle \zeta, \eta \rangle \,\mathrm{d\,vol}$$
for every $\zeta \in \mathcal{W}^{k,q}(\Sigma, \Lambda^{0,1}T^*\Sigma \otimes_{\mathscr{C}} E)$ then $\xi \in \mathcal{W}^{k,q}(\Sigma, E)$ and $D\xi = \eta$.

The proof is worked out in [2], Theorem C.2.3.

C.3 Proof of Riemann Roch theorem for S^2

The first part of Theorem C.1.3 follows from the theory of Theorem C.2.3. Therefore one just has to establish the index formula and prove the injectivity/surjectivity statements in (iii) for line bundles. Note that when proving the index formula it suffices to consider smooth complex linear operators. Namely, by Lemma C.2.2, every Cauchy-Riemann operator of class $\mathcal{W}^{\ell-1,p}$ differs from such an operator by a compact operator and so, by Theorem A.2.4, both operators have the same index. Moreover, by Theorem C.1.3 (i), it suffices to prove the index formula for $k = 1$ and $q = 2$. However, the statements in (iii) are more delicate and require one to consider the particular operators given. The proof is elaborate and is worked out in [2], Appendix C.4.

Here, we prove the baby version in the case $\Sigma = S^2$. We slightly generalize Corollary 2.3.3 to the following, using the identification $\ker D \equiv \mathrm{coker}\, D^*$, combined with the similarity principle.

Theorem C.3.1. Suppose $(E, J) \to S^2$ is a complex line bundle.

- If $c_1(E) < 0$, then D is injective.
- If $c_1(E) > -2$, then D is surjective.

Proof. The criterion for injectivity is an easy consequence of the similarity principle, for which we only need that D is a CR type operator. If $E \to S^2$ has complex rank 1 and $\ker D$ contains a nontrivial section η, then by the similarity principle, η has only isolated (and thus finitely many) zeroes, each of which counts with positive order. The count of these computes the first Chern number of E, thus $c_1(E) \geq 0$, and D must be injective if $c_1(E) < 0$.

The second part follows from the observation that D is surjective iff D^* is injective, and the latter is guaranteed by the condition $c_1(E^*) < 0$, which by Serre duality, is equivalent to $c_1(E) > -2$. □

The above already gives us enough information to deduce the index formula in the special case $\Sigma = S^2$, following [6] Theorem 3.4.1. Note that the index of a CR type operator $D : \mathcal{W}^{k,p}(E) \to \mathcal{W}^{k-1,p}(\Lambda^{0,1}T^*\Sigma \otimes E)$ depends only on the isomorphism class of the bundle $(E, J) \to (\Sigma, j)$. The difference between any two operators D and D' on the same bundle (as before) defines a smooth real-linear bundle map $A : E \to \Lambda^{0,1}T^*\Sigma \otimes E$ (often called the "zeroth order term"). It defines a bounded linear map from $\mathcal{W}^{k,p}(E)$ to $\mathcal{W}^{k,p}(\Lambda^{0,1}T^*\Sigma \otimes E)$, which is then composed with the compact inclusion into $\mathcal{W}^{k-1,p}(\Lambda^{0,1}T^*\Sigma \otimes E)$ and is therefore a compact operator. We conclude that all CR type operators on the same bundle are compact perturbations of each other, and thus have the same Fredholm index. Since the complex vector bundles over a closed surface are classified up to isomorphism by the first Chern number, the index will therefore depend only on the topological type of Σ and on $c_1(E)$. To compute it, we use the fact that every complex bundle admits a *complex-linear* CR operator (just take some complex connection ∇ on E, then $\nabla + J \circ \nabla \circ j$ is such an operator), and restrict our attention to the complex linear case. Then E is a holomorphic vector bundle, and $\ker D$ is simply the vector space of holomorphic sections. We employ this strategy in the following discussion, and construct model bundles with prescribed first Chern numbers.

Theorem C.3.2 (Riemann-Roch for S^2). $\operatorname{ind} D = 2n + 2c_1(E)$

Proof. We assume that $n = 1$. In this situation, at least one of the criteria $c_1(E) < 0$ or $c_1(E) > -2$ from the previous theorem is always satisfied, hence D is always injective or surjective; in fact, if $c_1(E) = -1$ it is an isomorphism. By considering D^* instead of D if necessary, we can restrict our attention to the case where D is surjective, so $\operatorname{ind} D = \dim\ker D$. We will now construct for each value of $c_1(E) \geq 0$ a "model" holomorphic line bundle, which is sufficiently simple so that we can identify the space of holomorphic sections explicitly.

For the case $c_1(E) = 0$, the model bundle is obvious: just take the trivial line bundle on S^2, so the holomorphic sections are constants, and therefore $\dim_{\mathbb{R}} \ker D = 2$. A more general model bundle can be defined by gluing together two local trivializations: let $E^{(1)}$ and $E^{(2)}$ denote two copies of the trivial holomorphic line bundle $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$, and for any $k \in \mathbb{Z}$, define

$$E_k := \left(E^{(1)} \coprod E^{(2)}\right) / (z, v) \sim \Phi_k(z, v),$$

where $\Phi_k : E^{(1)}|_{\mathbb{C}\setminus\{0\}} \to E^{(2)}|_{\mathbb{C}\setminus\{0\}}$ is a bundle isomorphism covering the biholomorphic map $z \mapsto 1/z$ and defined by $\Phi_k(z, v) = (1/z, g_k(z)v)$, with $g_k(z)v := v/z^k$. The function $g_k(z)$ is a holomorphic transition map, thus E_k has a natural holomorphic structure. Regarding a function $f : \mathbb{C} \to \mathbb{C}$ as a section of $E^{(1)}$, we have

$$\Phi_k(1/z, f(1/z)) = (z, z^k f(1/z)),$$

which means that f extends to a smooth section of E_k iff the function $g(z) = z^k f(1/z)$ extends smoothly to $z = 0$. It follows that $c_1(E_k) = k$, as one can choose $f(z) = 1$ for z in the unit disk and then modify $g(z) = z^k$ to a smooth function that algebraically has k zeroes at 0 (note that an actual modification is necessary only if $k < 0$). Similarly, the holomorphic sections of E_k can be identified with the entire functions $f : \mathbb{C} \to \mathbb{C}$ such that $z^k f(1/z)$ extends holomorphically to $z = 0$; if $k < 0$ this implies $f \equiv 0$, and if $k \geq 0$ it means $f(z)$ is a polynomial of degree at most k, hence $\dim\ker D = 2 + 2k$. The proof of the index theorem when $n = 1$ is now complete.

The case $n \geq 2$ can easily be derived from the above. It suffices to prove that $\operatorname{ind} D = 2n + 2c_1(E)$ for some model holomorphic bundle of rank n with a given value of $c_1(E)$. Indeed, for any $k \in \mathbb{Z}$, take E to be the direct sum of n holomorphic line bundles,

$$E := E_{-1} \oplus \cdots \oplus E_{-1} \oplus E_k,$$

which has $c_1(E) = k - (n-1)$. By construction, the natural CR operator D on E splits into a direct sum of CR operators on its summands, and it is an isomorphism on each of the E_{-1} factors, thus we conclude as in the line bundle case that D is injective if $k < 0$ and surjective if $k \geq 0$. By replacing D with D^* if necessary, we can now assume WLOG that $k \geq 0$ and D is surjective. The space of holomorphic sections is then simply the direct sum of the corresponding spaces for its summands, which are trivial for E_{-1} and have dimension $2 + 2k$ for E_k. We therefore have

$$\operatorname{ind} D = \dim\ker D = 2 + 2k = 2n + 2[k - (n-1)] = 2n + 2c_1(E).$$

□

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